Mathematics 5853 Name (please print) ________________

Examination II

November 9, 2004

Instructions: Give brief, clear answers.

I. (10) Prove that every compact subset of a Hausdorff space is closed.

Let C be a compact subset of a Hausdorff space X. Let $z \in X - C$. For each $x \in C$, choose disjoint open sets U_x and V_x with $x \in U_x$ and $z \in V_x$. The collection of all U_x contains C, so since C is compact, there is a finite subcollection with $C \subseteq U_{x_1} \cup \ldots \cup U_{x_n}$. Let $V = \cap_{i=1}^n V_{x_i}$. The V is an open neighborhood of z, and $V \cap C \subseteq V \cap (\cup_{i=1}^n U_{x_i}) = \cup_{i=1}^n (V \cap U_{x_i}) \subseteq \cup_{i=1}^n (V_{x_i} \cap U_{x_i}) = \emptyset$, that is, $z \in V \subseteq X - C$. Therefore $X - C$ is open.

II. (10) Define what it means to say that a space X is *locally compact.* Define the topology on the 1-point compactification $X^+ = X \cup \{\infty\}$, and prove that if X is locally compact Hausdorff, then X^+ is Hausdorff.

X is locally compact if for every $x \in X$, there exists a compact set C in X that contains an open neighborhood of x. A set U is open in X^+ when either (1) $U \subseteq X$ and U is open in X, or (2) $\infty \in U$ and $X^+ - U$ is compact. Suppose X is locally compact Hausdorff, and let $x, y \in X^+$ with $x \neq y$. If $x, y \in X$, then since X is Hausdorff there are disjoint open sets U and V in X with $x \in U$ and $y \in V$, and U and V are open in X^+ as well. If one of x or y, say y, equals ∞, then select a compact subset C in X that contains an open neighborhood U of x. Taking $V = X^+ - C$, we have that U and V are disjoint open sets in X^+ with $x \in U$ and $\infty \in V$.

III. (5) Let $\mathcal{U}$ be an open cover of a metric space (X, d). Define what it means to say that the number δ is a *Lebesgue number* for $\mathcal{U}$.

A number δ is a Lebesgue number for an open cover $\mathcal{U}$ of X if every subset of X of diameter less than δ is contained in some element of $\mathcal{U}$ (where the diameter of a subset A is defined to be the infimum of the distances between pairs of points in A).

IV. (10) Prove that if X is locally path-connected, then it has a basis that consists of path-connected sets.

Define $\mathcal{B}$ to be the collection of path-connected open subsets of X. The sets of $\mathcal{B}$ are open by definition. Suppose that $x \in X$ and U is an open neighborhood of x. Since X is locally connected, there exists a path-connected open neighborhood V of x with $x \in V \subseteq U$. By the Basis Recognition Theorem, $\mathcal{B}$ is a basis for the topology on X.

V. (10) Briefly describe the stereographic projection homeomorphism between $\mathbb{R}^2$ and $S^2 - \{(0,0,1)\}$ (formulas are not necessary, but a good picture is necessary). On a second picture of S^2, indicate the subsets of S^2 that correspond to the circles $x^2 + y^2 = n^2$ (for $n \in \mathbb{N}$) of $\mathbb{R}^2$, and indicate the subset of S^2 that corresponds to the x-axis of $\mathbb{R}^2$.

See last page.

VI. Let X be a connected metric space.
(10)

1. Suppose that the connected metric space (X, d) contains two points a and b with $d(a, b) = 2$. Prove that there exists a point $c \in X$ for which $d(a, c) = 1$. Hint: use the continuous function $D\colon X \to \mathbb{R}$ defined by $D(x) = d(a, x)$.
2. Prove that there exists a point $c \in X$ with $d(a, c) = d(b, c)$.
3. Show by example that there need not exist a point such that $d(a, c) = d(b, c) = 1$.

   1. Define $D\colon X \to \mathbb{R}$ by $D(x) = d(a, x)$ (D is continuous since it is the restriction of $d\colon X \times X \to \mathbb{R}$ to the subspace $\{a\} \times X$ of $X \times X$). We have $D(a) = 0$ and $D(b) = 2$. Since X is connected, the Intermediate Value Theorem implies that there exists $c \in X$ with $D(c) = 1$, that is, $d(a, c) = 1$.
   2. This time, define $f\colon X \to \mathbb{R}$ by $f(x) = d(a, x) - d(b, x)$. We have $f(a) = -2$ and $f(b) = 2$. Since X is connected, the Intermediate Value Theorem implies that there exists $c \in X$ with $f(c) = 0$, that is, $d(a, c) = d(b, c)$.
   3. In the unit circle S^1 in $\mathbb{R}^2$, the points $(1, 0)$ and $(-1, 0)$ are at distance 2, but the circle contains no point at distance 1 from both of these points.

VII. Let $(\mathbb{R}, \mathcal{L})$ be $\mathbb{R}$ with the lower-limit topology.
(10)

1. Prove that $(\mathbb{R}, \mathcal{L}) \times (\mathbb{R}, \mathcal{L})$ is separable. Hint: $\mathbb{Q} \times \mathbb{Q}$ is countable.
2. Find a subspace of $(\mathbb{R}, \mathcal{L}) \times (\mathbb{R}, \mathcal{L})$ that is not separable.

   1. $\mathbb{Q} \times \mathbb{Q}$ is countable. Let $\mathcal{B}$ be the basis for $(\mathbb{R}, \mathcal{L})$ consisting of all half-open intervals $[a, b)$. Then, the collection of all sets of the form $[a, b) \times [c, d)$ is a basis for the product topology on $(\mathbb{R}, \mathcal{L}) \times (\mathbb{R}, \mathcal{L})$. Let $[a, b) \times [c, d)$ be any one of these sets. Choose rational numbers $r \in (a, b)$ and $s \in (c, d)$. Then (r, s) is a point of $\mathbb{Q} \times \mathbb{Q}$ contained in $[a, b) \times [c, d)$. We have shown that every basis element contains a point of $\mathbb{Q} \times \mathbb{Q}$, so $\mathbb{Q} \times \mathbb{Q}$ is dense.
   2. Let $A = \{(x, -x) \mid x \in \mathbb{R}\} \subset (\mathbb{R}, \mathcal{L}) \times (\mathbb{R}, \mathcal{L})$. Each point $(x, -x)$ of A is open in the subspace topology, since $\{(x, -x) = A \cap [x, x + 1) \times [x, x + 1)$. So A is uncountable and has the discrete topology. Any countable subset of A is closed, so is not dense, and therefore A is not separable.

VIII. Prove that if X and Y are path-connected spaces, then $X \times Y$ is path-connected.
(10)

Let (x_0, y_0) and (x_1, y_1) be any two points of $X \times Y$. Since X and Y are path-connected, there exists paths $\alpha\colon I \to X$ from x_0 to x_1 and $\beta\colon I \to X$ from y_0 to y_1. Define $\beta\colon I \to X \times Y$ by $\beta(t) = (\alpha(t), \beta(t))$. It is continuous because its coordinate functions are α and β, which are continuous, and $\beta(0) = (x_0, y_0)$ and $\beta(1) = (x_1, y_1)$.

IX. Prove that every continuous map from $\mathbb{R}$ to $\mathbb{Q}$ is constant.
(10)

The image of $\mathbb{R}$ under any continuous map must be connected. Since the only connected subsets of $\mathbb{Q}$ are its points, the image of $\mathbb{R}$ under any continuous map must be a single point, that is, the map must be constant.

Alternatively, suppose $f\colon \mathbb{R} \to \mathbb{Q}$ is nonconstant, so $f(q_1) < f(q_2)$ for some $q_1, q_2 \in \mathbb{Q}$. Follow f by the inclusion to obtain $g\colon \mathbb{R} \to \mathbb{Q} \subset \mathbb{R}$. Choose an irrational number r with $f(q_1) < r < f(q_2)$. Since the domain $\mathbb{R}$ of g is connected, the Intermediate Value Theorem implies that there exists $x \in \mathbb{R}$ with $g(x) = r$, but this is impossible since $g(x)$ must be a rational number.

X. (25) Prove or give a counterexample to each of the following assertions.

1. Let (X, d) be a metric space with the property that for every $\epsilon > 0$, there is a finite covering of X by balls of radius ϵ. Then X is compact.

   False, for example the open unit interval $(0, 1)$ has a finite covering by balls of radius ϵ for any $\epsilon > 0$ (choose $n \in \mathbb{N}$ with $1/n < \epsilon$ and take the ϵ-balls centered at m/n, $m \in \mathbb{N}$ and $0 < m < n$.)

2. The cofinite topology on $\mathbb{R} \times \mathbb{R}$ equals the product topology $(\mathbb{R}, \text{cofinite}) \times (\mathbb{R}, \text{cofinite})$.

   False, for example the subset $\{0\} \times \mathbb{R}$ is a product of a two closed subsets of $(\mathbb{R}, \text{cofinite})$, so is closed in the product topology on $(\mathbb{R}, \text{cofinite}) \times (\mathbb{R}, \text{cofinite})$. But is is neither finite nor all of $\mathbb{R} \times \mathbb{R}$, so it is not closed in the cofinite topology on $\mathbb{R} \times \mathbb{R}$.

3. If there is a subspace A of X for which there exists an unbounded continuous function from A to $\mathbb{R}$, then there exists an unbounded continuous function from X to $\mathbb{R}$.

   False, for example the function $f(x) = 1/x$ is a continuous unbounded on the subspace $(0, 1]$ of $[0, 1]$, but $[0, 1]$ has no unbounded continuous function, because it is compact.

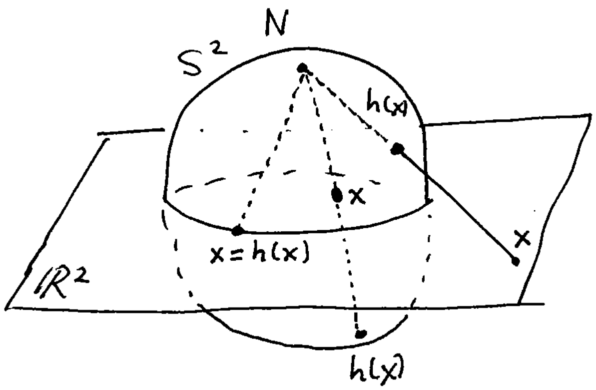
4. If every connected subspace of X is compact, then X is compact.

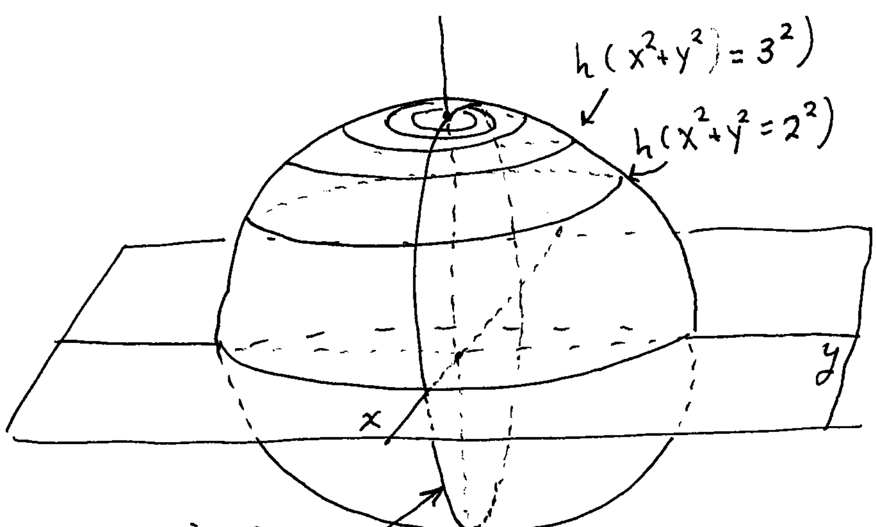
   False, for example every connected subspace of $\mathbb{Q}$ is a single point, so is compact, but $\mathbb{Q}$ is not compact.

5. If every compact subspace of X is connected, then X is connected.

   True. We will prove the contrapositive. Suppose X is not connected, and let $X = U \cup V$ be a separation. Since U and V are nonempty, we can choose points $x \in U$ and $y \in V$, and $x \neq y$ since $U \cap V = \emptyset$. The subspace $\{x, y\}$ is compact, since any finite space is compact, and has the discrete topology, since $U \cap \{x, y\} = \{x\}$ and $V \cap \{x, y\} = \{y\}$ are open, so $\{x, y\}$ is not connected. Therefore not every compact subspace of X is connected.

V. Briefly describe the stereographic projection homeomorphism between $\mathbb{R}^2$ and $S^2 - \{(0, 0, 1)\}$ (formulas are not necessary, but a good picture is necessary). On a second picture of S^2, indicate the subsets of S^2 that correspond to the circles $x^2 + y^2 = n^2$ (for $n \in \mathbb{N}$) of $\mathbb{R}^2$, and indicate the subset of S^2 that corresponds to the x-axis of $\mathbb{R}^2$.





h(x-axis) is a longitude circle

h(circle $x^2+y^2=n^2$) is a latitude circle, limiting to N as $n \to \infty$