I. Use a direct argument to prove the following implication: "The product of two odd integers is odd."

Let $m$ and $n$ be two odd integers. Since they are odd, we can write them as $m=2 k+1$ and $n=2 \ell+1$ for some integers $k$ and $\ell$. Then, we have $m n=(2 k+1)(2 \ell+1)=4 k \ell+2 \ell+2 k+1=2(2 k \ell+\ell+k)+1$, so $m n$ is odd.
II. Prove the following implication by proving its contrapositive: "If $n^{3} \not \equiv 2 \bmod 3$, then $n \not \equiv 2 \bmod 3$ "
(4) (where $\not \equiv$ means "is not congruent to").

We will argue the contrapositive. Assume that $n \equiv 2 \bmod 3$. Then $n^{3} \equiv 2^{3} \equiv 8 \equiv 2 \bmod 3$, the last congruence since $8-2$ is divisible by 3 .
III. Fill in the missing parts of the following proof by contradiction: [fill in] If $N$ is prime, then it is a prime (4) different from any of the $p_{i}$, a contradiction. If $N$ is composite, write it as $N=q_{1} q_{2} \cdots q_{m}$. [fill in]

Suppose for contradiction that there are only finitely many primes, say $p_{1}, p_{2}, \ldots, p_{k}$. Put $N=$ $p_{1} p_{2} \cdots p_{k}+1$. No $p_{i}$ divides $N$, since dividing $N$ by $p_{i}$ gives remainder 1 .
Then $q_{1}$ is a prime which divides $N$, so $q_{1}$ is a prime which is not equal to any of the $p_{i}$, again a contradiction.
IV. Disprove the following assertion: "For all integers $a, b$, and $n$, if $n \mid a b$, then $n \mid a$ or $n \mid b$."
$6 \mid 2 \cdot 3$, but $6 \nmid 2$ and $6 \nmid 3$.
V. Using the notation $G: X_{2} \rightarrow X_{1}$, define the range of $G$, the preimage of $x_{1}$ for an element $x_{1} \in X_{1}$, the (4) image of $x_{2}$ for an element $x_{2} \in X_{2}$, and the graph of $G$.

The range of $G$ is $\left\{y \in X_{1} \mid \exists x \in X_{2}, G(x)=y\right\}$, or $\left\{G(x) \mid x \in X_{2}\right\}$.
The preimage of $x_{1}$ is $\left\{x \in X_{2} \mid G(x)=x_{1}\right\}$.
The image of $x_{2}$ is $G\left(x_{2}\right)$.
The graph of $G$ is the set $\left\{(x, G(x)) \mid x \in X_{2}\right\}$ (or $\left.\left\{\left(x_{2}, x_{1}\right) \in X_{2} \times X_{1} \mid x_{1}=G\left(x_{2}\right)\right\}\right)$.
VI. Prove that the set $S$ of all two-element subsets of $\mathbb{N}$ is countable as follows: List all the two-element subsets that contain 1 as follows: $\{1,2\},\{1,3\},\{1,4\},\{1,5\} \ldots$ In a second row list all the subsets that contain 2 but do not contain 1 as: $\{2,3\},\{2,4\},\{2,5\},\{2,6\} \ldots$ Continue listing a third row and fourth row, so that if the process of writing rows were continued, every two-element subset would appear exactly once. Now describe how to form a single list in which every two-element subset appears exactly once.

Arrange the two-element subsets of $\mathbb{N}$ as follows:

$$
\begin{aligned}
& \{1,2\},\{1,3\},\{1,4\},\{1,5\}, \ldots \\
& \{2,3\},\{2,4\},\{2,5\},\{2,6\}, \ldots \\
& \{3,4\},\{3,5\},\{3,6\},\{3,7\}, \ldots \\
& \{4,5\},\{4,6\},\{4,7\},\{4,8\}, \ldots
\end{aligned}
$$

Each two-element subset appears exactly once; indeed, for $p<q,\{p, q\}$ is the $(q-p)^{\text {th }}$ entry in the $p^{\text {th }}$ row. The Cantor method of going up and down the diagonals enables us to turn this collection of lists into a single list: $\{1,2\},\{1,3\},\{2,3\},\{3,4\},\{2,4\},\{1,4\},\{1,5\},\{2,5\},\{3,5\},\{4,5\}, \ldots$ Then, we define a bijection from $\mathbb{N}$ to $S$ by sending $n$ to the $n^{t h}$ set in this list.
VII. Let $S$ be the set of functions from $\mathbb{N}$ to $\mathbb{N}$, that is, $S=\{f \mid f: \mathbb{N} \rightarrow \mathbb{N}\}$. Fill in the missing items in the following proof that $S$ is uncountable: "Suppose for [fill in] that $S$ is countable. Then, the elements of $\mathbb{N}$ can be listed as $f_{1}, f_{2}, \ldots$. Define a new function $g \in S$, that is, $g: \mathbb{N} \rightarrow \mathbb{N}$, as follows: For each $n \in \mathbb{N}$, let [fill in] if $f_{n}(n)=1$ and let [fill in] if $f_{n}(n) \neq 1$. Then for each $n, g(n) \neq f_{n}(n)$ so [fill in]. This is a contradiction, since [fill in]."

```
[ contradiction ]
\([g(n)=2]\)
\([g(n)=1]\)
\(\left[g \neq f_{n}\right]\)
[ every element of \(S\) is one of the \(f_{i}\) ]
```

VIII. Prove that $1 \cdot 1!+2 \cdot 2!+\cdots+n \cdot n!=(n+1)!-1$ whenever $n$ is a positive integer.

For $n=1$, we have $1 \cdot 1!=1 \cdot 1=1$ and $(1+1)!-1=2-1=1$, so the assertion is true for $n=1$. Inductively, assume that $1 \cdot 1!+2 \cdot 2!+\cdots+k \cdot k!=(k+1)!-1$. Then, $1 \cdot 1!+2 \cdot 2!+\cdots+k \cdot k!+(k+1) \cdot(k+1)!=$ $(k+1)!-1+(k+1) \cdot(k+1)!=(1+(k+1)) \cdot(k+1)!-1=(k+2) \cdot(k+1)!-1=(k+2)!-1$.
IX. State the Pigeonhole Principle (just the basic Pigeonhole Principle, not the Generalized Pigeonhole Prin(3) ciple).

$$
\text { If } k+1 \text { objects are place in } k \text { boxes, then at least one box contains at least two objects. }
$$

X. Fill in the missing parts of the following argument, which shows that if one selects any five different numbers
(4) between 2 and 9 , then some pair of them adds up to 11: "Let $S=\{2,3,4,5,6,7,8,9\}$, and select any five numbers from $S$. Define $f: S \rightarrow$ [fill in] by the rule $f(x)=x$ if $2 \leq x \leq 5$ and $f(x)=11-x$ if $6 \leq x \leq 9$. By the Pigeonhole Principle, there must be two of the five numbers, call them $m$ and $n$ with $m<n$, for which [fill in]. We cannot have both $m \leq 5$ and $n \leq 5$, since then we would have $m=f(m)=f(n)=n$. Nor can we have $m \geq 6$ and $n \geq 6$, since then we would have [fill in]. So we must have $m \leq 5$ and $n \geq 6$. But then, [fill in], so $m+n=11$."

$$
\begin{aligned}
& {[\{2,3,4,5\}]} \\
& {[f(m)=f(n)]} \\
& {[11-m=f(m)=f(n)=11-n \text { and therefore } m=n]} \\
& {[m=f(m)=f(n)=11-n,]}
\end{aligned}
$$

XI. This problem concerns strings of 10 digits $d_{1} d_{2} \cdots d_{10}$, where each $d_{i} \in\{0,1,2, \ldots, 9\}$. Leave the answers
(14) to the following questions as expressions which may contain factorials and/or binomial coefficients $\binom{m}{n}$, rather than calculating them out.

1. How many strings of 10 digits are there?

There are 10 possibilities for each digit in each of the 10 places, so by the Product Rule there are $10^{10}$ possibilities.
2. How many strings of 10 digits contain no 7 ?

There are 9 possibilities for each digit in each of the 10 places, so by the Product Rule there are $9^{10}$ possibilities.
3. How many strings of 10 digits start with three distinct digits?

There are 10 possibilities for the first digit, and for each of these there are then 9 for the second, then 8 for the third, then $10^{7}$ for the remaining seven digits. By the Product Rule, there are $10^{8} \cdot 9 \cdot 8$ possibilities.
4. How many strings of 10 digits have no two equal digits (that is, have all their digits different)?

This is exactly the number of permutations of 10 elements, 10 !.
5. How many strings of 10 digits contain exactly two 4's?

There are $\binom{10}{2}$ possible choices for where the two 4 's go, and for each of these choices there are $9^{8}$ possibilities for the remaining eight places. By the Product Rule, there are $\binom{10}{2} 9^{8}$ possibilities.
6. How many strings of 10 digits contain exactly two 4's and two 3 's?

There are $\binom{10}{2}$ possible choices for where the two 4's go. For each of these, there are $\binom{8}{2}$ possible choices for where the two 3 's go, and for each of these choices there are $8^{6}$ possibilities for the remaining six places. By the Product Rule, there are $\binom{10}{2}\binom{8}{2} 8^{6}$ possibilities.
7. How many strings of 10 digits either start with three 5 's or end with two 5 's, or both?

For a string starting with three 5 's, there are $10^{7}$ choices for the remaining seven places. For a string ending with two 5 's, there are $10^{8}$ choices for the remaining seven places. For a string both starting with three 5 's and ending with two 5 's, there are $10^{5}$ choices for the remaining five places. By the Inclusion-Exclusion Principle, the total number of possibilities is $10^{7}+10^{8}-10^{5}$.
XII. Draw Pascal's Triangle down to the row that contains the binomial coefficients $\binom{7}{k}$. Use this row and the (5) Binomial Theorem to write out $(a-1)^{7}$ (giving the explicit numerical values of the coefficients).


By the Binomial Theorem,

$$
\begin{gathered}
(a-1)^{7}=(a+(-1))^{7}=\sum_{k=0}^{7}\binom{7}{k} a^{7-k}(-1)^{k} \\
=\binom{7}{0} a^{7}(-1)^{0}+\binom{7}{1} a^{6}(-1)^{1}+\binom{7}{2} a^{5}(-1)^{2}+\binom{7}{3} a^{4}(-1)^{3} \\
+\binom{7}{4} a^{3}(-1)^{4}+\binom{7}{5} a^{2}(-1)^{5}+\binom{7}{6} a^{1}(-1)^{6}+\binom{7}{7} a^{0}(-1)^{7} \\
=a^{7}-7 a^{6}+21 a^{5}-35 a^{4}+35 a^{3}-21 a^{2}+7 a-1
\end{gathered}
$$

XIII. This problem concerns the identity $n\binom{n-1}{k-1}=k\binom{n}{k}$.

1. Verify the identity by calculation using the formula for $\binom{n-1}{k-1}$.

$$
\begin{aligned}
& n\binom{n-1}{k-1}=n \cdot \frac{(n-1)!}{(k-1)!((n-1)-(k-1))!}=\frac{n \cdot(n-1)!}{(k-1)!(n-k)!} \\
= & \frac{n!}{(k-1)!(n-k)!}=k \cdot \frac{n!}{k(k-1)!(n-k)!}=k \cdot \frac{n!}{k!(n-k)!}=k\binom{n}{k}
\end{aligned}
$$

2. Verify the identity by counting using two different methods the number of ways to choose a subset with $k$ elements from a set of $n$ elements, then choose one of the elements of this subset. That is, let $S$ be a set of $n$ elements and count the number of pairs $(A, x)$ where $A$ has $k$ elements, $A \subseteq S$, and $x \in A$.

There are $\binom{n}{k}$ ways to choose a subset $A$ of $k$ elements, and for each of these ways threre are $k$ ways to choose an element $x$ of $A$, giving $k\binom{n}{k}$ possible ways.
On the other hand, if we think of selecting $x$ first, there are $n$ possible ways to select $x$, and then there are $\binom{n-1}{k-1}$ possible ways to choose the remaining $k-1$ elements of $A$ from the remaining $n-1$ elements of $S$, giving $n\binom{n-1}{k-1}$ possible ways.
Since both of these methods counting the same thing, we must have $n\binom{n-1}{k-1}=k\binom{n}{k}$.
XIV. List all the ordered pairs in the relation on the set $A=\{1,2,3,4\}$ defined by $a \mathrm{R} b \Leftrightarrow a \mid b$.

$$
\begin{equation*}
\{(1,1),(1,2),(1,3),(1,4),(2,2),(2,4),(3,3),(4,4)\} \tag{3}
\end{equation*}
$$

XV. Verify that the relation of congruence modulo $m$ is an equivalence relation.

To prove that congruence is reflexive, let $a$ be any integer. Since $m \mid a-a$, we have $a \equiv a \bmod m$.
To prove that congruence is symmetric, assume that $a \equiv b \bmod m$. Then $m \mid a-b$, so $m \mid(-1)(a-b)$, $m \mid b-a$. Therefore $b \equiv a \bmod m$.
To prove that congruence is transitive, assume that $a \equiv b \bmod m$ and $b \equiv c \bmod m$. Then $m \mid a-b$ and $m \mid b-c$. Therefore $m \mid(a-b)+(b-c)$, which says that $m \mid a-c$. That is, $a \equiv c$ mod $m$.
XVI. List explicitly the elements in the congruence classes [0], [1], [2], [3], [4], [5] for the equivalence relation (6) of congruence modulo 6 on the set of integers (show at least five integers from each class). Write out the addition and multiplication tables for the set $\{[0],[1],[2],[3],[4],[5]\}$. Which elements have inverses for multiplication?

$$
\begin{aligned}
& {[0]=\{\ldots,-12,-6,0,6,12, \ldots\}} \\
& {[1]=\{\ldots,-11,-5,1,7,13, \ldots\}} \\
& {[2]=\{\ldots,-10,-4,2,8,14, \ldots\}} \\
& {[3]=\{\ldots,-9,-3,3,9,15, \ldots\}} \\
& {[4]=\{\ldots,-8,-2,4,10,16, \ldots\}} \\
& {[5]=\{\ldots,-7,-1,5,11,17, \ldots\}}
\end{aligned}
$$

$$
\begin{array}{cccccccccccc}
+ & {[0]} & {[1]} & {[2]} & {[3]} & {[4]} & {[5]} & \cdot & {[0]} & {[1]} & {[2]} & {[3]}
\end{array}[4] \quad[5]
$$

$$
\begin{array}{cccccccccccccc}
{[0]} & {[0]} & {[1]} & {[2]} & {[3]} & {[4]} & {[5]} & {[0]} & {[0]} & {[0]} & {[0]} & {[0]} & {[0]} & {[0]}
\end{array}
$$

| $[1]$ | $[1]$ | $[2]$ | $[3]$ | $[4]$ | $[5]$ | $[0]$ | $[1]$ | $[0]$ | $[1]$ | $[2]$ | $[3]$ | $[4]$ | $[5]$ |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |


| $[2]$ | $[2]$ | $[3]$ | $[4]$ | $[5]$ | $[0]$ | $[1]$ | $[2]$ | $[0]$ | $[2]$ | $[4]$ | $[0]$ | $[2]$ | $[4]$ |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $[3]$ | $[3]$ | $[4]$ | $[5]$ | $[0]$ | $[1]$ | $[2]$ | $[3]$ | $[0]$ | $[3]$ | $[0]$ | $[3]$ | $[0]$ | $[3]$ |
| $[4]$ | $[4]$ | $[5]$ | $[0]$ | $[1]$ | $[2]$ | $[3]$ | $[4]$ | $[0]$ | $[4]$ | $[2]$ | $[0]$ | $[4]$ | $[2]$ |
| $[5]$ | $[5]$ | $[0]$ | $[1]$ | $[2]$ | $[3]$ | $[4]$ | $[5]$ | $[0]$ | $[5]$ | $[4]$ | $[3]$ | $[2]$ | $[1]$ |

From the table, only [1] and [5] have multiplicative inverses (which is because 1 and 5 are the only two that are relatively prime to 6 ).

