Mathematics 2433-003H | Name (please print) ____________________

Final Examination

December 11, 2007

I. (7) Analyze the convergence behavior of the power series $\sum_{n=1}^{\infty} \frac{1}{nb^n}(x-a)^n$, where a and b are constants with $b > 0$. That is, determine its center, radius of convergence, and for every real number x determine whether the series converges absolutely, converges conditionally, or diverges.

II. (7) State the Comparison Test, and use it to verify that $\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$ diverges. (Hint: First verify that $\lim_{n\to\infty} n^{1/n} = 1$.)

III. (5) Graph the equation $r = \cos(\theta/3)$ for $0 \le \theta \le 6\pi$, first in the θ-r plane, then as a polar equation in the x-y plane.

IV. (7) State the Limit Comparison Test, and use it to verify that $\sum_{n=1}^{\infty}(\sqrt[n]{2}-1)$ diverges. (Hints: Use L'Hôpital's Rule to compare it to $\sum \frac{1}{n}$. You may need the facts that $\lim_{n\to\infty} 2^{1/n} = 1$ and $\frac{d(a^x)}{dx} = a^x \ln(a)$.)

V. (7) Give examples of the following:

1. A divergent series whose terms limit to 0.
2. A conditionally convergent series.
3. A geometric series $\sum_{n=0}^{\infty} r^n$ that converges to π.

VI. (6) Give examples of the following:

1. A power series $\sum_{n=0}^{\infty} c_n x^n$ that converges only for $x = 0$.
2. A power series $\sum_{n=0}^{\infty} c_n x^n$ whose radius of convergence is π.

VII. (3) Derive these formulas expressing rectangular coordinates in terms of spherical coordinates: $x = \rho\sin(\phi)\cos(\theta)$, $y = \rho\sin(\phi)\sin(\theta)$, $z = \rho\cos(\phi)$.

VIII. (4) In higher dimensions, say dimension n, there are vectors $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n$ that play the roles of $\vec{\imath}$, $\vec{\jmath}$, and $\vec{k}$. In particular, $\vec{e}_i \cdot \vec{e}_j = 0$ when $i \ne j$, and $\vec{e}_i \cdot \vec{e}_i = 1$ for each i. Verify that if an n-dimesional vector $\vec{v}$ equals $r_1\vec{e_1} + r_2\vec{e_2} + \cdots + r_n\vec{e_n}$, then $r_i = \vec{v} \cdot \vec{e}_i$ for each i.

IX. (6) Give examples of the following:

1. Vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ for which $(\vec{a} \times \vec{b}) \times \vec{c} \ne \vec{a} \times (\vec{b} \times \vec{c})$.
2. Nonzero vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ for which $\vec{a} \times \vec{b} = \vec{a} \times \vec{c}$ but $\vec{b} \ne \vec{c}$.

X. Find an equation for the plane that contains the points $(1, 2, 3)$, $(1, 3, 4)$, and $(2, 3, 5)$.
(5)

XI. A point moves according to the vector-valued function $\vec{r}(t) = e^t\vec{\imath} + e^{-t}\vec{\jmath}$.
(9)

1. Sketch the path of the point, indicating the direction of motion. (Hint: How are x and y related?)
2. Calculate the velocity vectors $\vec{r}\,'(t)$, the speed, and the unit tangent vector $\vec{T}(t)$.
3. Use $a_T = \dfrac{\vec{r}\,'(t)\cdot\vec{r}\,''(t)}{\|\,\vec{r}\,'(t)\,\|}$ and $a_N = \dfrac{\|\,\vec{r}\,'(t)\times\vec{r}\,''(t)\,\|}{\|\,\vec{r}\,'(t)\,\|}$ to calculate the tangential and normal components of the acceleration vector $\vec{a}(t)$.
4. When is the point speeding up? When is it slowing down?

XII. Write the general formula for the Taylor series of a function $f(x)$ at $x = a$. Use it to calculate the Taylor
(6) series of the function $f(x) = x^4$ at $x = 2$.

XIII. For the helix $\vec{r}(t) = 2\sin(t)\vec{\imath} + 3t\vec{\jmath} + 2\cos(t)\vec{k}$:
(8)

1. Calculate the unit tangent vector $\vec{T}(t)$, and use it to calculate the unit normal $\vec{N}(t)$.
2. Use the formula $\kappa = \dfrac{\|\,\vec{T}\,'(t)\,\|}{\|\,\vec{r}\,'(t)\,\|}$ to calculate the curvature.
3. Use the formula $\kappa = \left\|\dfrac{d\vec{T}}{ds}\right\|$ and the Chain Rule to calculate the curvature.

XIV. Bonus Problem: Let $u = 1 + \dfrac{x^3}{3!} + \dfrac{x^6}{6!} + \dfrac{x^9}{9!} + \cdots$, $v = x + \dfrac{x^4}{4!} + \dfrac{x^7}{7!} + \dfrac{x^{10}}{10!} + \cdots$, and $w = \dfrac{x^2}{2!} + \dfrac{x^5}{5!} + \dfrac{x^8}{8!} + \cdots$.
(6) Each of these converges by comparison with the Maclaurin Series of e^x. Show that $u^3 + v^3 + w^3 - 3uvw = 1$. (Hint: What is u'?)