Mathematics 2443-003 Name (please print) ____________________

Final Examination Form A

December 12, 2007

Instructions: Give brief, clear answers.

I. (4) Evaluate by changing to polar coordinates: $\displaystyle\iint_R x + y\, dR$ where R is the region between $x^2 + y^2 = 1$ and $x^2 + y^2 = 2$ and above the x-axis.

II. (6) For the function $f(x, y) = \ln(x^2+y^2)$, find the maximum rate of change at the point $(1, 2)$, and the direction in which it occurs. Find the directional derivative of f at $(1, 2)$ in the direction toward $(2, 4)$.

III. (6) Let S be the portion of the sphere of radius a that lies in the first octant. Use the standard parameterization of S to calculate $\displaystyle\iint_S (y\,\vec{\imath} - x\,\vec{\jmath} + \vec{k}\,) \cdot d\vec{S}$.

IV. (5) Use the Divergence Theorem to calculate the surface integral $\displaystyle\iint_S (x^2z^3\,\vec{\imath} + 2xyz^3\,\vec{\jmath} + xz^4\,\vec{k}\,) \cdot d\vec{S}$, where S is the surface of the box with $0 \le x \le 3$, $0 \le y \le 2$, $0 \le z \le 1$.

V. (4) The radius of a right circular cone is increasing at a rate of 6 in/s while its height is decreasing at a rate of 3 in/s. At what rate is the volume $V = \pi r^2 h/3$ changing when the radius is 10 and the height is 5?

VI. (6) Let S be the upper half of the sphere of radius 2, that is, the points (x, y, z) with $x^2 + y^2 + z^2 = 4$ and $z \ge 0$, and suppose that S is oriented with the upward normal. Use Stokes' Theorem to evaluate $\displaystyle\iint_S \operatorname{curl}(x^2e^{yz}\,\vec{\imath} + y^2e^{xz}\,\vec{\jmath} + z^2e^{xy}\,\vec{k}) \cdot d\vec{S}$.

VII. (4) Let S be the upper half of the sphere of radius 1, that is, the points (x, y, z) with $x^2 + y^2 + z^2 = 1$ and $z \ge 0$. Using the geometric interpretation of the surface integral of a vector field as the "flux" (that is, not by calculation using a parameterization or a formula from the formulas list), explain each of the following equalities:

1. $\displaystyle\iint_S \vec{\jmath} \cdot d\vec{S} = 0$
2. $\displaystyle\iint_S \vec{k} \cdot d\vec{S} = \pi$

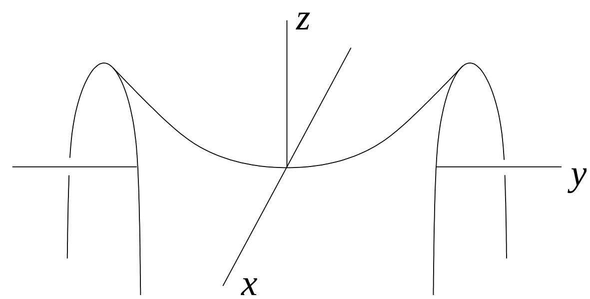
VIII. (4) Verify that the function $u = \cos(x - at) + \ln(x + at)$ is a solution to the wave equation $u_{tt} = a^2u_{xx}$.

IX. (10) Let S be the portion of the cylinder $x^2 + z^2 = 4$ that lies between the vertical planes $y = 0$ and $y = 2 - x$. The surface S is parameterized by $x = 2\cos(\theta)$, $y = h$, $z = 2\sin(\theta)$ for $0 \le \theta \le 2\pi$ and $0 \le h \le 2-2\cos(\theta)$.

1. Calculate $\vec{r}_\theta$, $\vec{r}_h$, $\vec{r}_h \times \vec{r}_\theta$, and $\|\,\vec{r}_h \times \vec{r}_\theta\,\|$.
2. Calculate $\displaystyle\iint_S x\, dS$.
3. Calculate $\displaystyle\iint_S x\,\vec{k} \cdot d\vec{S}$.

X. (6) The curl of the vector field $y\,\vec{\imath} - z\,\vec{\jmath} + x\,\vec{k}$ is $\vec{\imath} - \vec{\jmath} - \vec{k}$. Let S be the triangle which is the part of the plane $2x + y + z = 2$ that lies in the first octant. Give S the upward normal, and give its boundary C the corresponding positive orientation. Use Stokes' Theorem to evaluate the line integral $\displaystyle\int_C (y\,\vec{\imath} - z\,\vec{\jmath} + x\,\vec{k}) \cdot d\vec{r}$. (Hint: the surface integral on S is easy to calculate if one uses the definition $\displaystyle\iint_S \vec{G} \cdot d\vec{S} = \iint_S \vec{G} \cdot \vec{n}\, dS$.)

XI. (4) In an xy-coordinate system, sketch the gradient of the function whose graph is shown to the right.



XII. (3) A function R of the variables R_1, R_2, and R_3 is given implicitly by $\dfrac{1}{R} = \dfrac{1}{R_1} + \dfrac{1}{R_2} + \dfrac{1}{R_3}$. Use *implicit differentiation* to find $\dfrac{\partial R}{\partial R_3}$.

XIII. (5) Use the Divergence Theorem to show that if E is a solid with boundary the surface S, then $\displaystyle\iint_S \left(\frac{x}{3}\vec{\imath} + \frac{y}{3}\vec{\jmath} + \frac{z}{3}\vec{k}\right) \cdot d\vec{S}$ always equals the volume of E.

XIV. (6) On two different coordinate systems, graph the following vector fields:

1. $\vec{F}(x,y) = x\,\vec{\imath} + y\,\vec{\jmath}$
2. $\vec{F}(x,y) = \dfrac{-y}{x^2+y^2}\,\vec{\imath} + \dfrac{x}{x^2+y^2}\,\vec{\jmath}$

XV. (4) Sketch the region and change the order of integration for $\displaystyle\int_0^1 \int_{e^x}^{e} f(x,y)\,dy\,dx$.