Mathematics 2443-003 Name (please print) ____________________

Examination II Form A

October 18, 2007

Instructions: Give brief answers, but clearly indicate your reasoning.

I. (5) Evaluate the integral $\iint_R e^{y^2}\,dA$, where $R = \{(x,y) \mid 0 \le y \le 1,\ 0 \le x \le y\}$.

$$\iint_R e^{y^2}\,dA = \int_0^1 \int_0^y e^{y^2}\,dx\,dy = \int_0^1 xe^{y^2}\Big|_0^y\,dy = \int_0^1 ye^{y^2}\,dy = e^{y^2}/2\Big|_0^1 = \frac{e-1}{2}\ .$$

II. (9) Let E be the upper hemisphere of the unit ball, that is, $E = \{(x,y,z) \mid x^2+y^2+z^2 \le 1,\ z \ge 0\}$. For the integral $\iiint_E f(x,y,z)\ dV$, supply the explicit limits of integration, the expression for dV, and (if necessary) the expressions for x, y, and z, that would be needed to calculate the integral:

(i) In xyz-coordinates (x,y,z)

$\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} f(x,y,z)\,dz\,dy\,dx$

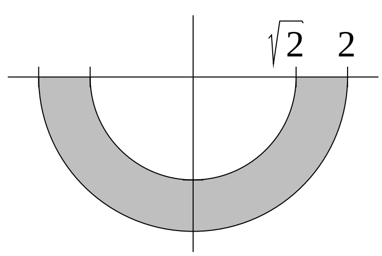
(ii) In cylindrical coordinates (r,θ,z)

$\int_0^{2\pi} \int_0^1 \int_0^{\sqrt{1-r^2}} f(r\cos(\theta), r\sin(\theta), z)\,dz\,dr\,d\theta$

(iii) In spherical coordinates (ρ,θ,ϕ)

$\int_0^{\pi/2} \int_0^{2\pi} \int_0^1 f(\rho\sin(\phi)\cos(\theta), \rho\sin(\phi)\sin(\theta), \rho\cos(\phi))\,d\rho\,d\theta\,d\phi$

III. (5) Evaluate by changing to polar coordinates: $\iint_R (x+y)\,dA$, where R is the region that lies below the x-axis and between the circles $x^2+y^2=2$ and $x^2+y^2=4$.



$$\iint_R (x+y)\,dA = \int_\pi^{2\pi} \int_{\sqrt{2}}^2 (r\cos(\theta) + r\sin(\theta))\,r\,dr\,d\theta = \int_\pi^{2\pi} \frac{r^3}{3}(\cos(\theta)+\sin(\theta))\Big|_{\sqrt{2}}^2\,d\theta$$

$$= \int_\pi^{2\pi} \frac{8-2\sqrt{2}}{3}(\cos(\theta)+\sin(\theta))\,d\theta = \frac{8-2\sqrt{2}}{3}(\sin(\theta)-\cos(\theta))\Big|_\pi^{2\pi} = \frac{4\sqrt{2}-16}{3}$$

IV. (5) Let E be the solid in the first octant bounded by $x^2+y^2+z^2=1$ and the three coordinate planes (that is, E is the portion of the unit ball that lies in the first octant). Suppose that the density at each point of E equals the distance from the point to the xz-plane. Write integrals to find the mass of E and its moment with respect to the yz-plane. Do not supply explicit limits for the integrals, or try to evaluate the integrals.

The density is $\rho(x,y,z) = y$. The mass and moment are $m = \iiint_E dm = \iiint_E y\ dV$, $M_{yz} = \iiint_E z\,dm = \iiint_E yz\ dV$.

V. (5) Sketch a portion of a typical graph $z = f(x, y)$, showing the tangent plane at a point $(x_0, y_0, f(x_0, y_0))$. Let $\vec{v}_y$ be the vector in the tangent plane whose $\vec{\imath}$-component is 0 and whose $\vec{\jmath}$-component is 1 (i. e. $\vec{v}_y$ is a vector of the form $\vec{\jmath} + \lambda\vec{k}$ for some number λ). Show $\vec{v}_y$ in your sketch, and express λ in terms of f or its partial derivatives.

For the sketch, see your class notes. λ is $f_y(x_0, y_0)$, so $\vec{v}_x = \vec{\jmath} + f_y(x_0, y_0)\vec{k}$.

VI. (5) Calculate $\| \, (\vec{\imath} + f_x(x_0, y_0)\vec{k}) \times (\vec{\jmath} + f_y(x_0, y_0)\vec{k}) \, \|$. Give the details of the calculation, not just the answer.

$$\| \, (\vec{\imath} + f_x(x_0, y_0)\vec{k}) \times (\vec{\jmath} + f_y(x_0, y_0)\vec{k}) \, \| = \left| \begin{matrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 1 & 0 & f_x(x_0, y_0) \\ 0 & 1 & f_y(x_0, y_0) \end{matrix} \right| = \| - f_x(x_0, y_0)\vec{\imath} - f_y(x_0, y_0)\vec{\jmath} + \vec{k} \|$$

$$= \sqrt{(-f_x(x_0, y_0))^2 + (-f_y(x_0, y_0))^2 + 1} = \sqrt{1 + f_x(x_0, y_0)^2 + f_y(x_0, y_0)^2}$$

VII. (6) Find the surface area of the portion of the paraboloid $z = x^2 + y^2$ that lies above the unit disk in the xy-plane.

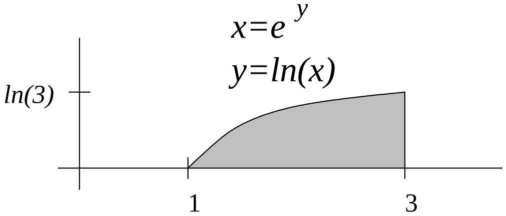
We calculate $dS = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dx\, dy = \sqrt{1 + 4x^2 + 4y^2}\, dx\, dy$. Integrating in polar coordinates, the surface area is

$$\int_0^{2\pi} \int_0^1 \sqrt{1 + 4r^2}\, r\, dr\, d\theta = \int_0^{2\pi} d\theta \left(\frac{2}{3} \frac{(1 + 4r^2)^{3/2}}{8} \Big|_0^1 \right) = \pi \, \frac{5\sqrt{5} - 1}{6} \, .$$

VIII. (5) Calculate the numerical value of a Riemann sum to estimate the value of $\iint_R xy^2 \, dA$, where R is the rectangle $[0, 4] \times [0, 2]$, i. e. the (x, y) with $0 \le x \le 4$ and $0 \le y \le 2$. Partition the x-interval $[0, 4]$ into two equal subintervals, and partition the y-interval into two equal subintervals, so that the Riemann sum has four terms. Use the Midpoint Rule to choose the sample points.

The rectangles are $[0, 2] \times [0, 1]$, $[2, 4] \times [0, 1]$, $[0, 2] \times [1, 2]$, and $[2, 4] \times [1, 2]$, and the corresponding midpoints are $(1, 1/2)$, $(3, 1/2)$, $(1, 3/2)$, and $(3, 3/2)$. The function values at the midpoints are $1/4$, $3/4$, $9/4$, and $27/4$. Since the area of each rectangle is 2, the Riemann sum is $(1/4) \cdot 2 + (3/4) \cdot 2 + (9/4) \cdot 2 + (27/4) \cdot 2 = 20$.

IX. (5) Sketch the region and change the order of integration for $\displaystyle\int_1^3 \int_0^{\ln(x)} f(x, y)\, dy\, dx$.



$$\int_0^{\ln(3)} \int_{e^y}^3 f(x, y)\, dx\, dy.$$

X. (5) Let E be the solid tetrahedron bounded by the coordinate planes and the plane $x + y + 2z = 2$. Supply limits for the integral $\iiint_E f(x, y, z)\, dV$, assuming that the order of integration is first with respect to y, then with respect to x, then with respect to z.

The top plane is $y = 2 - x - 2z$, and the side in the xz-plane (i. e. where $y = 0$) is the triangle bounded by the coordinate axes and the line $x + 2z = 2$. So the integral is $\int_0^1 \int_0^{2-2z} \int_0^{2-x-2z} f(x, y, z)\, dy\, dx\, dz$.

XI. (5) Evaluate the integral $\displaystyle\int_{-a}^{a} \int_0^{\sqrt{a^2-y^2}} (x^2 + y^2)^{3/2}\, dx\, dy$.

The domain of integration is the right half of the disk of radius a. Changing to polar coordinates, the integral becomes

$$\int_{-\pi/2}^{\pi/2} \int_0^a r^3 \cdot r\, dr\, d\theta = \int_{-\pi/2}^{\pi/2} \frac{a^5}{5}\, d\theta = \frac{\pi a^5}{5}\ .$$