

Instructions: Give brief answers, but clearly indicate your reasoning.

$$x = \rho \cos(\theta) \sin(\phi), y = \rho \sin(\theta) \sin(\phi), z = \rho \cos(\phi), dV = \rho^2 \sin(\phi) d\rho d\phi d\theta, \vec{r}_\phi \times \vec{r}_\theta = a \sin(\phi)(x\vec{i} + y\vec{j} + z\vec{k}),$$
$$\|\vec{r}_\phi \times \vec{r}_\theta\| = a^2 \sin(\phi)$$

$$dS = \sqrt{1 + g_x^2 + g_y^2} dD$$

$$dS = \|\vec{r}_u \times \vec{r}_v\| dD$$

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{n} dS$$

$$\iint_S (P\vec{i} + Q\vec{j} + R\vec{k}) \cdot d\vec{S} = \iint_D -P g_x - Q g_y + R dD$$

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F} \cdot (\vec{r}_u \times \vec{r}_v) dD$$

I. A path C is parameterized as a vector-valued function by $\vec{r}(t) = t\vec{i} + t^2\vec{j}$, $1 \leq t \leq 2$. Using this parameterization, evaluate the following line integrals.

1. $\int_C (y/x) dx$

2. $\int_C (y/x) ds$

II. Let $\vec{F}(x, y, z) = 2xy\vec{i} + (x^2 + 2yz)\vec{j} + (y^2 + 3z)\vec{k}$.

(6) 1. Find a function f such that $\vec{F} = \nabla f$.

2. Calculate $\int_C \vec{F} \cdot d\vec{r}$, where C is given by the parameterization $x = \sqrt{\cos^3(t)}$, $y = \cos^4(t)$, $z = \cos^5(t)$, $0 \leq t \leq \pi/2$.

III. Let $\vec{F}(x, y)$ be the vector field $\frac{-y}{x^2 + y^2}\vec{i} + \frac{x}{x^2 + y^2}\vec{j}$. Verify by calculation that $\int_C \vec{F} \cdot d\vec{r}$ is not path-independent on the domain $\{(x, y) \mid (x, y) \neq (0, 0)\}$. (Hint: Consider the line integral of \vec{F} on the unit circle C).

IV. Verify that if $P(x, y, z)\vec{i} + Q(x, y, z)\vec{j} + R(x, y, z)\vec{k}$ is conservative, then $\frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}$. (Hint: if it is conservative, then it can be written in the form $f_x\vec{i} + f_y\vec{j} + f_z\vec{k}$.)

V. Suppose that C is a closed loop with no self intersections, bounding a region D .

- (5)
1. Explain how one determines the “positive” or “standard” orientation on C .
 2. State Green’s Theorem.

VI. Calculate the curl and the divergence of the vector field $\vec{F}(x, y, z) = 3z^2\vec{i} + x \cos(y)\vec{j} - 2xz\vec{k}$.

(5)

VII. Let S be the portion of the cylinder $x^2 + z^2 = 1$ that lies between the vertical planes $y = 0$ and $y = 2 - x$.

(5) The surface S is parameterized by $x = \cos(\theta)$, $y = h$, $z = \sin(\theta)$ for $0 \leq \theta \leq 2\pi$ and $0 \leq h \leq 2 - \cos(\theta)$.

1. Calculate \vec{r}_θ and \vec{r}_h .

2. Calculate $\vec{r}_h \times \vec{r}_\theta$ and $\|\vec{r}_h \times \vec{r}_\theta\|$.

VIII. Use Green's Theorem to calculate $\int_C (y^3 \vec{i} - x^3 \vec{j}) \cdot d\vec{r}$, where C is the circle $x^2 + y^2 = 4$ with the clockwise orientation.
(6)

IX. Calculate $\iint_S (xy \vec{i} + 4x^2 \vec{j} + yz \vec{k}) \cdot d\vec{S}$, where S is the surface $z = xe^y$, $0 \leq x \leq 1$, $0 \leq y \leq 2$.
(6)

X. Let S be the part of the cone $z = \sqrt{x^2 + y^2}$ that lies between the planes $z = 1$ and $z = 2$. Calculate dS in terms of dD , where D is the domain in the xy -plane lying beneath S , and use it to calculate $\iint_S z^2 dS$.
(6)