Mathematics 2443-003 Name (please print) ____________________

Examination III Form B

November 29, 2007

Instructions: Give brief answers, but clearly indicate your reasoning.

$x = \rho\cos(\theta)\sin(\phi)$, $y = \rho\sin(\theta)\sin(\phi)$, $z = \rho\cos(\phi)$, $dV = \rho^2\sin(\phi)\,d\rho\,d\phi\,d\theta$, $\vec{r}_\phi \times \vec{r}_\theta = a\,\sin(\phi)(x\vec{\imath} + y\vec{\jmath} + z\vec{k})$,
$\|\,\vec{r}_\phi \times \vec{r}_\theta\,\| = a^2\,\sin(\phi)$
$dS = \sqrt{1 + g_x^2 + g_y^2}\;dD$
$dS = \|\,\vec{r}_u \times \vec{r}_v\,\|\,dD$
$\iint_S\,\vec{F}\cdot d\vec{S} = \iint_S\,\vec{F}\cdot\vec{n}\,dS$
$\iint_S\,(P\,\vec{\imath} + Q\,\vec{\jmath} + R\,\vec{k})\cdot d\vec{S} = \iint_D\,-P\,g_x - Q\,g_y + R\,dD$
$\iint_S\,\vec{F}\cdot d\vec{S} = \iint_D\,\vec{F}\cdot(\vec{r}_u \times \vec{r}_v)\,dD$

I. (6) A path C is parameterized as a vector-valued function by $\vec{r}(t) = t^2\vec{\imath} + t\vec{\jmath}$, $1 \le t \le 2$. Using this parameterization, evaluate the following line integrals.

1. $\int_C (x/y)\,dx$
2. $\int_C (x/y)\,ds$

II. (6) Let $\vec{F}(x,y,z) = 2xy\,\vec{\imath} + (x^2 + 2yz)\,\vec{\jmath} + (y^2 + z)\,\vec{k}$.

1. Find a function f such that $\vec{F} = \nabla f$.
2. Calculate $\int_C \vec{F}\cdot d\vec{r}$, where C is given by the parameterization $x = \cos^3(t)$, $y = \cos^4(t)$, $z = \sqrt{\cos(t)}$, $0 \le t \le \pi/2$.

III. (5) Let $\vec{F}(x,y)$ be the vector field $\dfrac{-y}{x^2+y^2}\vec{\imath} + \dfrac{x}{x^2+y^2}\vec{\jmath}$. Verify by calculation that $\int_C \vec{F}\cdot d\vec{r}$ is not path-independent on the domain $\{(x,y) \mid (x,y) \ne (0,0)\}$. (Hint: Consider the line integral of $\vec{F}$ on the unit circle C).

IV. (4) Verify that if $P(x,y,z)\,\vec{\imath} + Q(x,y,z)\,\vec{\jmath} + R(x,y,z)\,\vec{k}$ is conservative, then $\dfrac{\partial Q}{\partial z} = \dfrac{\partial R}{\partial y}$. (Hint: if it is conservative, then it can be written in the form $f_x\,\vec{\imath} + f_y\,\vec{\jmath} + f_z\,\vec{k}$.)

V. (5) Suppose that C is a closed loop with no self intersections, bounding a region D.

1. Explain how one determines the "positive" or "standard" orientation on C.
2. State Green's Theorem.

VI. (5) Calculate the curl and the divergence of the vector field $\vec{F}(x,y,z) = 3z^2\,\vec{\imath} - x\cos(y)\,\vec{\jmath} + 2xz\,\vec{k}$.

VII. (5) Let S be the portion of the cylinder $x^2 + z^2 = 1$ that lies between the vertical planes $y = 0$ and $y = 2 - x$. The surface S is parameterized by $x = \cos(\theta)$, $y = h$, $z = \sin(\theta)$ for $0 \le \theta \le 2\pi$ and $0 \le h \le 2 - \cos(\theta)$.

1. Calculate $\vec{r}_\theta$ and $\vec{r}_h$.
2. Calculate $\vec{r}_h \times \vec{r}_\theta$ and $\|\,\vec{r}_h \times \vec{r}_\theta\,\|$.

VIII. (6) Let S be the part of the cone $z = \sqrt{x^2 + y^2}$ that lies between the planes $z = 1$ and $z = 2$. Calculate dS in terms of dD, where D is the domain in the xy-plane lying beneath S, and use it to calculate $\iint_S z^2\, dS$.

IX. (6) Calculate $\iint_S (xy\,\vec{\imath} + 4x^2\,\vec{\jmath} + yz\,\vec{k}) \cdot d\vec{S}$, where S is the surface $z = xe^y$, $0 \le x \le 1$, $0 \le y \le 2$.

X. (6) Use Green's Theorem to calculate $\int_C (y^3\,\vec{\imath} - x^3\,\vec{\jmath}) \cdot d\vec{r}$, where C is the circle $x^2 + y^2 = 4$ with the clockwise orientation.