Mathematics 6813-001 Name (please print) ______________________

Final Examination

December 18, 2008

Instructions: Insofar as possible, give brief, clear answers. Use major theorems when possible.

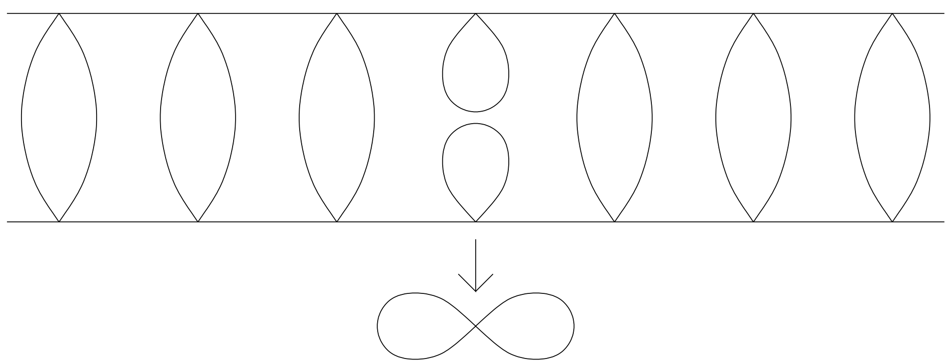
I. (7) Let $p\colon (\widetilde{X},\widetilde{x}) \to (X,x)$ be a covering map, and let α be a loop in X based at x. Let $\widetilde{\alpha}$ be the lift of α starting at $\widetilde{x}$. Prove that $\widetilde{\alpha}$ is a loop if and only if $[\alpha] \in p_\#(\pi_1(\widetilde{X},\widetilde{x}))$.

II. (7) Recall the proof that a path-connected, locally path-connected, semilocally simply connected space X has a simply-connected covering space $\widetilde{X}$. Tell how the points of $\widetilde{X}$ are defined, how the covering map $p\colon \widetilde{X} \to X$ is defined, and how the basic open sets in its topology are defined. You do not need to give any more details about the proof.

III. (8) Define the following: Δ^n, a *singular n-simplex*, $C_n(X)$, $C_n(X,A)$, a *chain map*, $f_\#$, f_*. Show how the fact that $f_\#$ is a chain map proves that f_* is well-defined.

IV. (6) State the *Homotopy Extension Property*. Use the fact that a subcomplex of a CW-complex has the HEP to prove the following proposition: Let A be a subcomplex of a CW-complex X. Suppose that $f\colon A \to Y$ is a continuous map that extends to a continuous map $F\colon X \to Y$. Suppose further that $f \simeq g$. Then g extends to a continuous map $G\colon X \to Y$.

V. (6) The figure to the right shows a certain covering space of the one-point union of two circles a and b.



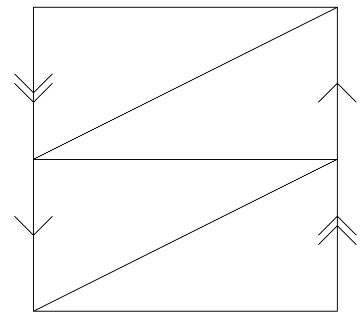
(i) Label a and b with single and double arrows. Make a corresponding labeling of the covering space that indicates a particular covering map.

(ii) Here is a sloppy way to state the Lifting Criterion: Let $p\colon \widetilde{X} \to X$ be a covering map, and let $f\colon Y \to X$ be a continuous map. Then f lifts to a map $F\colon Y \to \widetilde{X}$ with $pF = f$ if and only if $f_\#(\pi_1(Y)) \subseteq p_\#(\pi_1(\widetilde{X}))$. Give a precise statement of the Lifting Criterion, taking basepoints into account.

(iii) Use the example of a covering map given in part (i) to explain why basepoints must be taken into account in stating the Lifting Criterion.

VI. (6) Recall that the cone on a space A is the quotient space $CA = (A \times I)/(A \times \{1\})$. Let $A \subset X$, with A and X path-connected, and consider the quotient space $Y = X \cup CA$ obtained from X and CA by identifying each $(a,0) \in CA$ with $a \in A \subset X$. Let P be the cone point $[A \times \{1\}]$. Observe that $CA - (A \times \{0\})$ is contractible, and $Y - P$ deformation retracts to X (you do not need to give any argument, except drawing reasonable pictures). Use van Kampen's Theorem to give a description of $\pi_1(Y, y_0)$ at a basepoint y_0 in $A \times (0,1)$. (You can be a bit informal, but try to stay close to the statement of van Kampen's Theorem.)

VII. (11) The figure to the right shows a Δ-structure on a Möbius band X; the right and left sides of the square are identified as indicated to form the band. The Δ-structure has four 2-simplices, seven 1-simplices, and three 0-simplices. The top and bottom horizontal 1-simplices t and b form the boundary circle in X. The middle horizontal 1-simplex m has its endpoints identified and forms the "core circle" C of X. Orient t, m, and b from left to right. It is easy to check that X deformation retracts to C (you do not need to prove this), so that the inclusion $i_*\colon H_k(C) \to H_k(X)$ is an isomorphism for each k.



(i) The core circle C has a Δ-structure with one 1-simplex m and one 0-simplex v. Use this to calculate the homology of C. Since X deformation retracts to C, the inclusion $C \to X$ is an isomorphism on homology groups.

(ii) The boundary circle D of M has a Δ-structure with two 1-simplices t and b and two 0-simplices x and y, the left and right endpoints of t. Use this Δ-structure to calculate the homology of D.

(iii) Label orientations on the four 2-simplices τ_1, τ_2, τ_3, and τ_4 and on t, m, and b so that the 2-chain $c = \tau_1 + \tau_2 + \tau_3 + \tau_4$ has $\partial c = t + b - 2m$.

(iv) Use the chain in part (iii) (even if you did not find it explicitly) to explain why the inclusion $j\colon D \to X$ carries a generator of $H_1(D)$ to $2[m] \in H_1(X)$.

(v) Deduce that X does not retract to D.

VIII. (5) Let F and G be chain maps from the chain complex $\cdots \to A_{n+1} \xrightarrow{\partial} A_n \xrightarrow{\partial} A_{n-1} \to \cdots$ to the chain complex $\cdots \to B_{n+1} \xrightarrow{\partial} B_n \xrightarrow{\partial} B_{n-1} \to \cdots$. Define a *chain homotopy* from F to G. Verify that if P is a chain homotopy from F to G, then $F_* = G_*\colon H_n(A) \to H_n(B)$.

IX. (8) Consider a commutative diagram of abelian groups and homomorphisms:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{i} & B & \xrightarrow{j} & C & \longrightarrow & 0 \\ \downarrow & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow \\ 0 & \longrightarrow & A' & \xrightarrow{i'} & B' & \xrightarrow{j'} & C' & \longrightarrow & 0 \end{array}$$

with exact rows.

(i) Prove that if α and γ are injective, then so is β.

(ii) Prove that if α and γ are surjective, then so is β.

X. (11) Let X be the one-point union of two circles. For each of the following groups G, display a 4-fold covering space of X with deck transformation group G: $\{1\}$, C_2, $C_2 \times C_2$, C_4 (you do not need to verify that those are the deck transformation groups). Display an infinite-sheeted covering space of X with fundamental group $\mathbb{Z}$. Again, it is not necessary to verify that it is a covering, but use the single and double arrow method to clarify what the covering map is.