Mathematics 2423-001 Name (please print) ____________________

Exam III

November 16, 2011

Instructions: Give concise answers, but clearly indicate your reasoning. It is *not* expected that you will be able to answer all the questions, just do whatever you can in 50 minutes.

I. (4) Find the point on the graph of $y = \cosh(x)$ where the tangent line has slope 2. Express the answer as an expression involving the logarithm funtion, not an inverse hyperbolic trig function.

We want the point where $\cosh'(x) = \sinh(x) = 2$, or $(e^x - x^{-2})/2 = 2$. Solving for x, we have $e^x - e^{-x} = 4$, $(e^x)^2 - 4e^x - 1 = 0$, and $e^x = 2 \pm \sqrt{5}$. Since $e^x > 0$, the only possibility is $e^x = 2 + \sqrt{5}$, so $x = \ln(2 + \sqrt{5})$.

II. (16) Evaluate the following integrals.

1. $\int \tan^3(x)\, dx$

$$\int \tan^3(x)\, dx = \int \tan(x)(\sec^2(x) - 1)\, dx = \int \tan(x)\sec^2(x) - \tan(x)\, dx = \frac{\tan^2(x)}{2} + \ln|\cos(x)| + C$$

2. $\displaystyle\int \frac{\log_{10}(x)}{x}\, dx$

$$\int \frac{\log_{10}(x)}{x}\, dx = \int \frac{\ln(x)}{\ln(10)x}\, dx = \int \frac{1}{\ln(10)} \ln(x)\, d(\ln(x)) = \frac{\ln(x)^2}{2\ln(10)} + C$$

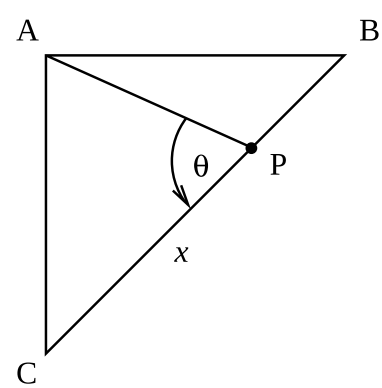
3. $\int x^3 e^{-x^2}\, dx$. You may make use of the fact that $\int x e^{-x^2}\, dx = -e^{-x^2}/2 + C$.

Using integration by parts with $u = x^2$, $du = 2x\, dx$, $dv = xe^{-x^2}\, dx$, $v = -e^{-x^2}/2$, we have $\int x^3 e^{-x^2}\, dx = -x^2 e^{-x^2}/2 + \int xe^{-x^2}\, dx = -x^2 e^{-x^2}/2 - e^{-x^2}/2 + C = -e^{-x^2}(1 + x^2)/2 + C$.

4. $\displaystyle\int \frac{1}{\sqrt{x}(1 + x)}\, dx$

Putting $u = \sqrt{x}$, $du = \frac{1}{2\sqrt{x}}\, dx$, we have $\displaystyle\int \frac{1}{\sqrt{x}(1+x)}\, dx = \int \frac{2}{1 + u^2}\, du = 2\tan^{-1}(\sqrt{x}) + C$.

III. (4) The triangle ABC is an isosceles right triangle whose legs AB and AC each have length 2. Let $0 < x \leq 2\sqrt{2}$ be the distance from C to the point P on the hypotenuse of ABC. Express the angle θ as a function of x. (Hint: draw the horizontal line from P to AC.)



Let P' be the point where the horizontal line from P to AC meets AC, and let ω be the angle $P'PA$, so that $\theta = \pi/4 + \omega$. Since CP has length x, PP' and $P'C$ each have length $x/\sqrt{2}$, and consequently $P'A$ has length $2 - x/\sqrt{2}$. Since the length of $P'A$ divided by the length of $P'P$ is $\tan(\omega)$, and $\pi/4 \leq \omega < \pi/2$, we have $\omega = \tan^{-1}\left(\frac{2 - x/\sqrt{2}}{x/\sqrt{2}}\right) = \tan^{-1}\left(\frac{2\sqrt{2} - x}{x}\right)$, and therefore $\theta = \pi/4 + \tan^{-1}\left(\frac{2\sqrt{2} - x}{x}\right)$.

IV. Evaluate the following limits.
(12)

1. $\lim_{x\to\infty} x\tan(1/x)$

   Using l'Hôpital's rule, $\lim_{x\to\infty} x\tan(1/x) = \lim_{x\to\infty} \frac{\tan(1/x)}{1/x} = \lim_{x\to\infty} \frac{-\sec^2(1/x)/x^2}{-1/x^2} = \lim_{x\to\infty} \sec^2(1/x) = \sec^2(0) = 1$.

2. $\lim_{x\to 0}(1-2x)^{1/x}$

   We have $\lim_{x\to 0}(1-2x)^{1/x} = \lim_{x\to 0} e^{\ln((1-2x)^{1/x})} = \lim_{x\to 0} e^{\ln(1-2x)/x}$. Using l'Hôpital's rule, $\lim_{x\to 0}\frac{\ln(1-2x)}{x} = \lim_{x\to 0}\frac{-2/(1-2x)}{1} = -2$, so $\lim_{x\to 0}(1-2x)^{1/x} = e^{-2}$.

3. $\lim_{x\to 0} \frac{1}{x^3}\int_0^x \sin\left(\frac{1}{2}\pi t^2\right)\,dt$

   Using l'Hôpital's rule and the FTC,

$$\lim_{x\to 0} \frac{1}{x^3}\int_0^x \sin\left(\frac{1}{2}\pi t^2\right)\,dt = \lim_{x\to 0}\frac{\int_0^x \sin\left(\frac{1}{2}\pi t^2\right)\,dt}{x^3} = \lim_{x\to 0}\frac{\sin\left(\frac{1}{2}\pi x^2\right)}{3x^2}$$
$$= \lim_{x\to 0}\frac{\pi x\cos\left(\frac{1}{2}\pi x^2\right)}{6x} = \lim_{x\to 0}\frac{\pi\cos\left(\frac{1}{2}\pi x^2\right)}{6} = \frac{\pi}{6}$$

   (Or alternatively $\lim_{x\to 0}\frac{\sin\left(\frac{1}{2}\pi x^2\right)}{3x^2} = \lim_{x\to 0}\frac{\pi}{6}\,\frac{\sin\left(\frac{1}{2}\pi x^2\right)}{\frac{1}{2}\pi x^2} = \frac{\pi}{6}$.)

V. Define what it means to say that a function f is *injective* (also called *one-to-one*). Assuming that f is an
(4) injective function with domain A and range B, define what it means to say that a function g is the *inverse* of f.

f is *injective* when $f(x_1) = f(x_2)$ implies that $x_1 = x_2$.

g is the *inverse* of f if the domain of g is B, the range of g is A, and $g(x) = y$ exactly when $f(y) = x$.

VI. Obtain the reduction formula $\int (\ln(x))^n\,dx = x(\ln(x))^n - n\int (\ln(x))^{n-1}\,dx$.
(4)

Using integration by parts with $u = \ln(x)^n$, $du = n\ln(x)^{n-1}/x\,dx$, $dv = dx$, and $v = x$, we have $\int (\ln(x))^n\,dx = x(\ln(x))^n - \int x\cdot n\ln(x)^{n-1}/x\,dx = x(\ln(x))^n - n\int (\ln(x))^{n-1}\,dx$.

VII. Find the exact values of arctan(tan(1)) and arctan(tan(6)),
(3)

1 radian is between $-\pi/2$ and $\pi/2$, so $\arctan(\tan(1)) = 1$.
6 is just a little less than 2π, so an angle of 6 radians lies in quadrant IV (where $x > 0$ and $y < 0$). To obtain an angle with the same tangent but lying between $-\pi/2$ and $\pi/2$, we must subtract 2π. So $\arctan(\tan(6)) = \arctan(\tan(6-2\pi)) = 6 - 2\pi/2$.

VIII. Calculate the following derivatives.
(6)

1. $\dfrac{d}{dx}\log_{10}(\ln(x))$

$$\frac{d}{dx}\log_{10}(\ln(x)) = \frac{d}{dx}\frac{\ln(\ln(x))}{\ln(10)} = \frac{\frac{1}{\ln(x)}\cdot\frac{1}{x}}{\ln(10)} = \frac{1}{x\ln(x)\ln(10)}$$

2. $\dfrac{d}{dx}10^{\ln(x)}$

$$\frac{d}{dx}10^{\ln(x)} = \frac{d}{dx}e^{\ln(x)\ln(10)} = (e^{\ln(x)\ln(10)})\cdot\frac{\ln(10)}{x} = \frac{10^{\ln(x)}\ln(10)}{x}$$