Mathematics 2934-010 Name (please print) ____________________

Exam II

October 17, 2011

Instructions: Give concise answers, but clearly indicate your reasoning. Do not simplify the answers unless you find yourself with enough time at the end.

I. Calculate the following.
(12)

1. $\dfrac{\partial}{\partial y}\left(\displaystyle\int_y^{x^2} e^{-t^2}\,dt\right)$

   $\dfrac{\partial}{\partial y}\left(\displaystyle\int_y^{x^2} e^{-t^2}\,dt\right) = \dfrac{\partial}{\partial y}\left(-\displaystyle\int_{x^2}^{y} e^{-t^2}\,dt\right) = e^{-y^2}$, using the Fundamental Theorem of Calculus.

2. $\dfrac{\partial}{\partial x}\left(\displaystyle\int_y^{x^2} e^{-t^2}\,dt\right)$

   $\dfrac{\partial}{\partial x}\left(\displaystyle\int_y^{x^2} e^{-t^2}\,dt\right) = 2xe^{-(x^2)^2} = 2xe^{-x^4}$, using the Fundamental Theorem of Calculus and the Chain Rule.

3. $f_{tx}(x, y, z, t)$ if $f_x(x, y, z, t) = \dfrac{1}{x^2y^2z^2t^2}$

   Using Clairaut's Theorem, $f_{tx}(x, y, z, t) = (f_x)_t(x, y, z, t) = -\dfrac{1}{(x^2y^2z^2t^2)^2}\,(2tx^2y^2z^2) = -\dfrac{2}{x^2y^2z^2t^3}$.

4. df if $f(x, y, z) = x^3yz$

   $3x^2yz\,dx + x^3z\,dy + x^3y\,dz$.

II. Clairaut's Theorem tells us there is no function $g(x, y)$ for which $\nabla g = 2x\sin(y)\vec{\imath} + ye^x\vec{\jmath}$. Explain why.
(4)

If we have such a g, then $g_x = 2x\sin(y)$ and $g_y = ye^x$. We would then have $g_{xy} = 2x\cos(y)$ and $g_{yx} = ye^x$, but this would violate Clairaut's Theorem, which says that $g_{xy} = g_{yx}$.

III. Let $f(x, y, z) = x^2 + y^2 + z^3$.
(8)

1. Find the maximum directional rate of change at $(x, y, z) = (-2, 0, 1)$.

   We have $\nabla f = 2x\vec{\imath} + 2y\vec{\jmath} + 3z^2\vec{k}$, so $\nabla f(-2, 0, 1) = -4\vec{\imath} + 3\vec{k}$. The maximum rate of change is $\|\nabla f(-1, -1, -2)\| = \|-4\vec{\imath} + 3\vec{k}\| = \sqrt{25} = 5$.

2. Find the rate of change at $(x, y, z) = (-1, -1, -2)$ in the direction $2\vec{\imath} - \vec{\jmath} - 2\vec{k}$.

   We have $\nabla f(-1, -1, -2) = -2\vec{\imath} - 2\vec{\jmath} + 12\vec{k}$. A unit vector in the direction of $2\vec{\imath} - jj - 2\vec{k}$ is $\vec{u} = (2\vec{\imath} - \vec{\jmath} - 2\vec{k})/\|2\vec{\imath} - \vec{\jmath} - 2\vec{k}\| = \frac{2}{3}\vec{\imath} - \frac{1}{3}\vec{\jmath} - \frac{2}{3}\vec{k}$, so the rate of change is $\nabla f(-1, -1, -2) \cdot \vec{u} = -\frac{26}{3}$.

3. Write an equation for the tangent plane to the level surface through $(-2, 0, 1)$.

   $\nabla f(-2, 0, 1) = -4\vec{\imath} + 3\vec{k}$ is a normal vector to the tangent plane, so an equation for the tangent plane is $-4(x + 2) + 3(z - 1) = 0$.

IV. Let $f(x, y)$ be a differentiable function of x and y.
(7)

(a) Use the Chain Rule to calculate $\dfrac{\partial f}{\partial \theta}$, where θ is the polar angle. Express $\dfrac{\partial f}{\partial \theta}$ purely in terms of x, y, $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ (that is, without using r or θ).

$$\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial}{\partial \theta}(r\cos(\theta)) + \frac{\partial f}{\partial y}\frac{\partial}{\partial \theta}(r\sin(\theta)) = \frac{\partial f}{\partial x}(-r\sin(\theta)) + \frac{\partial f}{\partial y}r\cos(\theta) = -y\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial y}.$$

(b) In the case of $f(x, y) = x^2 + y^2$, the formula should give $\dfrac{\partial f}{\partial \theta} = 0$. Why is the fact that $\dfrac{\partial}{\partial \theta}(x^2 + y^2) = 0$ geometrically obvious without doing any calculations at all?

The level curves are circles centered at the origin, so moving in the θ direction is moving tangent to the level curves. So the rate of change as one increases θ is 0.

V. (a) Define the *graph* of a function $f(x, y)$ of two variables.
(5)

It is the set of points in 3-dimensional space of the form $(x, y, f(x, y))$. [The key word here is *set,* since it tells the type of object that a graph is. An answer like "It is the graph of $z = f(x, y)$" merits partial credit, but it is to some extent begging the question by defining the graph of f to be the graph of a certain equation, without explicitly saying what a graph (of either one) is.]

(b) For a certain function $f(x, y)$, $f(1, 4) = 5$ and $\dfrac{\partial f}{\partial y}(1, 4) = 3$. Let L be the line tangent to the graph of f at the point $(1, 4, 5)$ and lying in the plane $x = 1$. Tell a direction vector for L.

A direction vector for L is the vector we called $\vec{v}_y$, that is tangent to the graph and points in the y-direction. In this case, $\vec{v}_y = \vec{\jmath} + f_y(1, 4)\vec{k} = \vec{\jmath} + 3\vec{k}$. [The information that $f(1, 4) = 5$ is not needed, although it would be needed to write an equation for L.]

VI. Suppose that $\vec{T}(t)$ is the unit tangent vector to some curve. Show that $\vec{T}'(t)$ is orthogonal (that is,
(3) perpendicular) to $\vec{T}(t)$.

$\dfrac{d}{dt}(\vec{T}(t)\cdot\vec{T}(t)) = \dfrac{d}{dt}(1) = 0$, but also $\dfrac{d}{dt}(\vec{T}(t)\cdot\vec{T}(t)) = \vec{T}'(t)\cdot\vec{T}(t) + \vec{T}(t)\cdot\vec{T}'(t) = \vec{T}'(t)\cdot\vec{T}(t) + \vec{T}'(t)\cdot\vec{T}(t) = 2(\vec{T}'(t)\cdot\vec{T}(t))$, so $= \vec{T}'(t)\cdot\vec{T}(t) = 0$ and consequently $\vec{T}'(t)$ is orthogonal to $\vec{T}(t)$.

VII. For the vector-valued function $\vec{r}(t) = 4\cos(t/2)\vec{\imath} + 4\sin(t/2)\vec{\jmath}$, the velocity is $\vec{v}(t) = -2\sin(t/2)\vec{\imath} + 2\cos(t/2)\vec{\jmath}$.
(10)

(a) Draw a large graph showing the curve traced out by $\vec{r}(t)$ for $0 \le t \le 3\pi$.

(The curve is the portion of a circle of radius 4 between $\theta = 0$ and $\theta = 3\pi/2$.)

(b) Calculate the velocity and speed, and draw the velocity vectors for $t = \pi$ and $t = 1$, showing with reasonable accuracy their locations and lengths.

The speed is $\|\vec{v}(t)\| = 2$.

For $t = \pi$ the point of tangency is $(0, 4)$ and $\vec{v}(\pi) = -2\vec{\imath}$ points left and has length half the radius of the circle.

$\vec{v}(1) = -2\sin(1/2) + 2\cos(1/2)\vec{\imath}$. Since 1 radian is about $\pi/3$, 1/2 radian is about $\pi/6$, and $\vec{v}(1)$ is tangent where the angle is around $\pi/6$, has length 2 and points northwest.

(c) Calculate $\|\vec{T}'(t)\|/(ds/dt)$.

Dividing $\vec{v}(t)$ by the speed 2 gives the speed $\vec{v}(t)/\|\vec{v}(t)\| = \vec{T}(t) = -\sin(t/2)\vec{\imath} + \cos(t/2)\vec{\jmath}$, so $\vec{T}'(t) = -\cos(t/2)/2\,\vec{\imath} - \sin(t/2)/2\,\vec{\jmath}$. Its length is $1/2$, so the curvature is $(1/2)/2 = 1/4$ [which we know is the curvature of a circle of radius 4].

(d) Calculate the tangential component of the vectors $\vec{w}(t) = -3\vec{\imath}$.

We just take the scalar product with the unit vector in the tangential direction, $(-3\vec{\imath}) \cdot \vec{T}(t) = -3(-\sin(t/2)) = 3\sin(t/2)$.

VIII. A particle moves with acceleration $2t\,\vec{\imath} - 12t\,\vec{\jmath}$. At time $t = 0$, it is located at $(0, 1, 0)$ and is moving with
(4) velocity $\vec{k}$. Find its position function $\vec{r}(t)$.

Integrating, we have $\vec{v}(t) = (t^2 + C_1)\vec{\imath} + (6t^2 + C_2)\vec{\jmath} + C_3\vec{k}$. Since $\vec{k} = \vec{v}(0) = C_1\vec{\imath} + C_2\vec{\jmath} + C_3\vec{k}$, we have $C_1 = C_2 = 0$ and $C_3 = 1$, so $\vec{v}(t) = t^2\vec{\imath} + 6t^2\vec{\jmath} + \vec{k}$.

Integrating again gives $\vec{r}(t) = (t^3/3 + C_1)\vec{\imath} + (2t^3 + C_2)\vec{\jmath} + (t + C_3)\vec{k}$, and from $\vec{\jmath} = \vec{r}(0)$ we find $C_1 = C_3 = 0$ and $C_2 = 1$. Therefore $\vec{r}(t) = t^3/2\,\vec{\imath} + (2t^3 + 1)\vec{\jmath} + t\,\vec{k}$.