Mathematics 2934-001 Name (please print) ______________________

Exam III

November 14, 2011

Instructions: Give concise answers, but clearly indicate your reasoning.

I. (5) Find all critical points of the function $e^y(y^2 - x^2)$.

We have $\dfrac{\partial}{\partial x} e^y(y^2-x^2) = -2xe^y$ and $\dfrac{\partial}{\partial y} e^y(y^2-x^2) = e^y(y^2-x^2) + 2e^y y = e^y(y^2+2y-x^2)$. The first is zero only when $x=0$. When $x=0$, the second is zero only when $y^2+2y=0$, that is, when $y=0$ or $y=-2$. So the critical points are $(0,0)$ and $(0,-2)$.

II. (4) Here is a minimization problem: "Find three numbers whose sum is 18 and so that the sum of the square of the first, plus twice the square of the second, plus three times the square of the third, is as small as possible." Set up the minimization problem as a problem of minimizing a function of two variables, but do not proceed further with the problem.

Call two of the numbers x and y, so the third is $18-x-y$. The function described in the problem is $f(x,y) = x^2 + 2y^2 + 3(18-x-y)^2$.

III. (6) Find the maximum and minimum values of the function $f(x,y) = x^2+y$ on the unit circle, by parameterizing the boundary and expressing f as a function of the parameter, then finding the maximum and minimum values of that function.

We parameterize the unit circle as $x = \cos(\theta)$, $y = \sin(\theta)$, so on the unit circle the function is $g(\theta) = f(\cos(\theta), \sin(\theta)) = \cos^2(\theta) + \sin(\theta)$.

Seeking critical points, we have $g'(\theta) = -2\cos(\theta)\sin(\theta) + \cos(\theta) = \cos(\theta)(1-2\sin(\theta))$. This is zero when $\cos(\theta) = 0$, which occurs at $\theta = \pi/2$ and $\theta = 3\pi/2$, or when $\sin(\theta) = 1/2$, which occurs at $\theta = \pi/6$ and $\theta = 5\pi/6$.

The corresponding four points on the circle are $(0,1)$, $(0,-1)$, $(\sqrt{3}/2, 1/2)$ and $(-\sqrt{3}/2, 1/2)$.

Since $f(0,1) = 1$, $f(0,-1) = -1$, $f(\sqrt{3}/2, 1/2) = 5/4$ and $f(-\sqrt{3}/2, 1/2) = 5/4$, the maximimu is $5/4$ at $(\sqrt{3}/2, 1/2)$ and $(-\sqrt{3}/2, 1/2)$, and the minimum is -1 at $(0,-1)$.

IV. (6) For the integral $\displaystyle\int_{-3}^{0}\int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} x^3 + xy^2\,dx\,dy$:

(a) Rewrite the integral to reverse the order of integration and supply the limits of integration, but do not continue further with its evaluation.

The domain of integration is the portion of the disk of radius 3 at the origin that lies below the x-axis. Since the bottom circle has equation $y = -\sqrt{9-x^2}$, the rewritten integral is $\displaystyle\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{0} x^3+xy^2\,dy\,dx$.

(b) Rewrite the integral using polar coordinates and supply the limits of integration, but do not continue further with its evaluation.

In polar coordinates the region is $0 \le r \le 3$, $\pi \le \theta \le 2\pi$. The integrand becomes $x(x^2+y^2) = r\cos(\theta)r^2$, and $dy\,dx$ becomes $r\,dr\,d\theta$, so the rewritten integral is $\displaystyle\int_{\pi}^{2\pi}\int_{0}^{3} r^4\cos(\theta)\,dr\,d\theta$.

V. (6) Consider a lamina occupying the region D in the xy-plane which is a triangle with vertices $(0,0)$, $(1,2)$, and $(2,0)$. Suppose that its density at the point (x,y) is $x+y$. Write expressions involving integrals for each of the following. Supply limits of integration for the integrals, but do not evaluate them.

(a) The mass of the lamina.

The mass is $\iint_D x + y\, dA$. The bottom side of D is $y = 0$, the upper left side is $y = 2x$, and the upper right side is $y = 4 - 2x$. It looks easier to integrate with respect to x first, using the expressions $x = y/2$ and $x = 2 - y/2$, so the integral can be written as $\int_0^2 \int_{y/2}^{2-y/2} x + y\, dx\, dy$.

Integrating with respect to y first is acceptable, giving the more complicated expression $\int_0^1 \int_0^{2x} x + y\, dy\, dx + \int_1^2 \int_0^{4-2x} x + y\, dy\, dx$.

(b) The x-coordinate of its center of mass.

The distance from the y-axis is x, so we have

$$\overline{x} = \frac{M_y}{m} = \frac{\int_0^2 \int_{y/2}^{2-y/2} x^2 + xy\, dx\, dy}{\int_0^2 \int_{y/2}^{2-y/2} x + y\, dx\, dy}$$

or, integrating with respect to y first,

$$\frac{\int_0^1 \int_0^{2x} x^2 + xy\, dy\, dx + \int_1^2 \int_0^{4-2x} x^2 + xy\, dy\, dx}{\int_0^1 \int_0^{2x} x + y\, dy\, dx + \int_1^2 \int_0^{4-2x} x + y\, dy\, dx}$$

VI. (8) Recall that if a surface S is the graph of a function f, that is, when it is the graph of the equation $z = f(x, y)$, then the local relation between surface area on S and surface area $dR = dx\, dy$ in the xy-plane below it is that $dS = \sqrt{1 + f_x^2 + f_y^2}\, dR$. Let S be the portion of the sphere $x^2 + y^2 + z^2 = 3$ that lies inside the region above the cone $z = \sqrt{x^2 + y^2}$.

(a) Compute $\sqrt{1 + f_x^2 + f_y^2}$ for the function $f(x, y) = \sqrt{3 - x^2 - y^2}$.

$$\sqrt{1 + f_x^2 + f_y^2} = \sqrt{1 + \left(\frac{-x}{\sqrt{3 - x^2 - y^2}}\right)^2 + \left(\frac{-y}{\sqrt{3 - x^2 - y^2}}\right)^2}$$
$$= \sqrt{1 + \frac{x^2}{3 - x^2 - y^2} + \frac{y^2}{3 - x^2 - y^2}} = \sqrt{\frac{3}{3 - x^2 - y^2}} = \frac{\sqrt{3}}{\sqrt{3 - x^2 - y^2}}$$

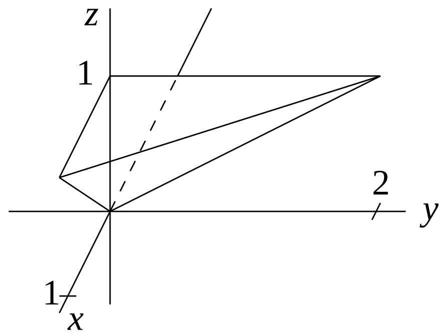
(b) Find the region R in the xy-plane that lies below the surface S.

The top half of the sphere is $z = \sqrt{3 - x^2 - y^2}$. The intersection of the sphere with the cone is where $\sqrt{3 - x^2 - y^2} = \sqrt{x^2 + y^2}$, that is, where $x^2 + y^2 = 3/2$, the circle of radius $\sqrt{3/2}$, and R is the disk bounded by this circle.

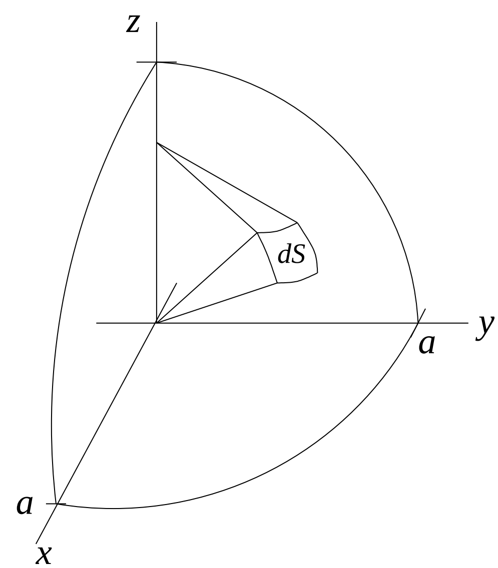
(c) Write an integral whose value is the area of S. Express the integral in polar coordinates and supply the limits of integration, but do not evaluate the integral.

$$\iint_R \frac{\sqrt{3}}{\sqrt{3 - x^2 - y^2}}\, dR = \int_0^{2\pi} \int_0^{\sqrt{3/2}} \frac{\sqrt{3}r}{\sqrt{3 - r^2}}\, dr\, d\theta\ .$$

VII. (4) Sketch the solid whose volume is given by the integral $\int_0^1 \int_0^{2-2x} \int_{x+\frac{y}{2}}^1 dz\,dy\,dx$. It might help to notice that $z = x + \frac{y}{2}$ is a plane that passes through $(0, 2, 1)$ and $(1, 0, 1)$.



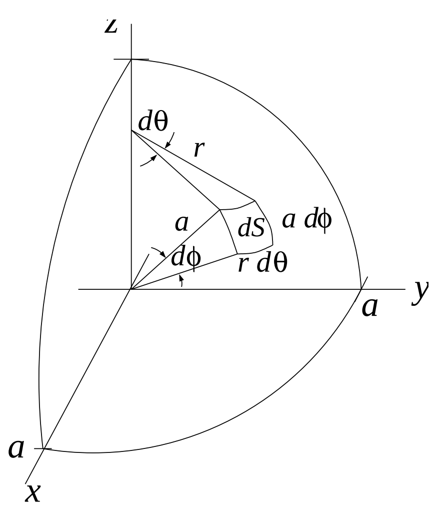
VIII. (7) When we specialize spherical coordinates to the points on a sphere of radius a, we obtain a parameterization of the sphere in terms of θ and ϕ, given by the formulas $x = a\sin(\phi)\cos(\theta)$, $y = a\sin(\phi)\sin(\theta)$, $z = a\cos(\phi)$. There is also the useful formula $r = a\sin(\phi)$.



(a) Draw the region in the $\theta\phi$-plane that corresponds to the entire surface of the sphere. Show which lines correspond to latitudes on the sphere, and which lines corrspond to longitudes on the sphere.

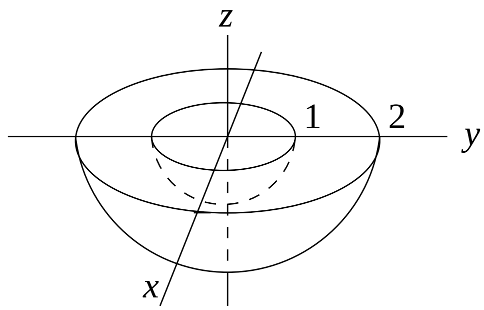
It is the rectangle $0 \le \theta \le 2\pi$, $0 \le \phi \le \pi$. The horizontal lines correspond to latitudes, and the vertical lines correspond to longitudes.

(b) In the figure to the right, label $d\theta$ and $d\phi$. Label various distances and use them to figure out an expression for dS in terms of $d\theta$ and $d\phi$.



Since $r = a\sin(\phi)$, we have $dS = r\,d\theta\,a\,d\phi = a\sin(\phi)\,d\theta\,a\,d\phi = a^2\sin(\phi)\,d\theta\,d\phi$.

IX. Sketch the solid whose volume is given by the integral $\displaystyle\int_{\pi/2}^{\pi}\int_0^{2\pi}\int_1^2 \rho^2\sin(\phi)\,d\rho\,d\theta\,d\phi$.
(4)



X. Consider the transformation $x = e^{s-t}$, $y = e^{s+t}$ from the st-plane to the region $x > 0$, $y > 0$ in the xy-plane.
(10)

(a) Calculate the Jacobian of this parameterization.

$$\begin{vmatrix} x_s & y_s \\ x_t & y_t \end{vmatrix} = \begin{vmatrix} e^{s-t} & e^{s+t} \\ -e^{s-t} & e^{s+t} \end{vmatrix} = e^{s-t}e^{s+t} + e^{s+t}e^{s-t} = 2e^{2s} \ .$$

(b) Find the (s,t)-values where the transformation increases area locally.

It increases area locally at the points where $2e^{2s} > 1$. Solving this, we obtain

$$\begin{aligned} e^{2s} &> 1/2 \\ 2s &> \ln(1/2) = -\ln(2) \\ s &> -\ln(2)/2 \end{aligned}$$

(c) Let R be the region in the xy-plane bounded by $xy = 1$, $xy = 3$, $y = x$, and $y = 3x$, as in the problem that we analyzed in class. Take as given the fact that R corresponds to the square S in the st-plane given by $0 \le s \le \ln(3)/2$ and $0 \le t \le \ln(3)/2$, write an integral in terms of s and t which equals $\displaystyle\iint_R xy\,dA$. Supply the limits of integration, but do not evaluate the integral (the evaluation is not hard, it gives the answer $2\ln(3)$ which agrees with the two methods we used for this problem in class).

$$\iint_R xy\,dA = \iint_S e^{s-t}e^{s+t}\cdot 2e^{2s}\,dS = \int_0^{\ln(3)/2}\int_0^{\ln(3)/2} 2e^{4s}\,ds\,dt$$