RECURSIVE ENUMERATION OF PYTHAGOREAN TRIPLES

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In [**9**], P. W. Wade and W. R. Wade (no relation to the second author) gave a recursion formula that produces Pythagorean triples. In fact, it produces all Pythagorean triples (a, b, c) having a given value of the *height,* which is defined to be $h = c - b$. For the cases when h is a square or twice a square, they gave a complete proof that the recursion generates all Pythagorean triples. In this note, we give a quick proof of this for all values of h, using a formula that gives all Pythagorean triples.

We call the formula the *height-excess* enumeration because its parameters are the height and certain multiples of the *excess* $e = a + b - c$. This enumeration method appears several times in the literature, but does not seem to be widely known. We will discuss these origins after giving the formula. A more extensive treatment of the height-excess enumeration and other applications of it appear in [**7**].

To set terminology, a *Pythagorean triple* (PT) is an ordered triple (a, b, c) of positive integers such that $a^2 + b^2 = c^2$. A PT is *primitive* when it is not a multiple of a smaller triple. A PT with $a < b$ is called a *Pythagorean triangle.* A number is called *square-free* if it is not divisible by the square of any prime number.

Height-Excess Enumeration Theorem. *To a positive integer h, written as pq^2 with p square-free and q positive, associate the number d equal to $2pq$ if p is odd, and to pq if p is even. As one takes all pairs (h, k) of positive integers, the formula*

$$P(k,h) = \left(h + dk, dk + \frac{(dk)^2}{2h}, h + dk + \frac{(dk)^2}{2h}\right)$$

produces each Pythagorean triple exactly once. The primitive Pythagorean triples occur exactly when $\gcd(k, h) = 1$ and either $h = q^2$ with q odd, or $h = 2q^2$. The Pythagorean triangles occur exactly when $k > \frac{h}{d}\sqrt{2}$.

Notice that h is the height and dk is the excess of $P(k, h)$. Figure 1 shows geometric interpretations of h and k. The number d is called the *increment.*

Table 1 shows the first 12 PT's of height h when h is 1, 2, 81, and 162, with the primitive PT's indicated by an asterisk.

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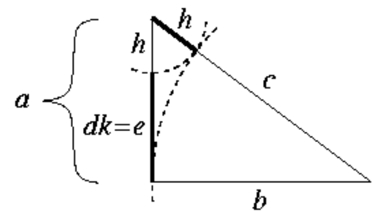

FIGURE 1. Height and excess of a Pythagorean triangle.

$h=1$	$h=2$	$h=81$	$h=162$
$(3,4,5)$*	$(4,3,5)$*	$(99,20,101)$*	$(180,19,181)$*
$(5,12,13)$*	$(6,8,10)$	$(117,44,125)$*	$(198,40,202)$
$(7,24,25)$*	$(8,15,17)$*	$(135,72,153)$	$(216,63,225)$
$(9,40,41)$*	$(10,24,26)$	$(153,104,185)$*	$(234,88,250)$
$(11,60,61)$*	$(12,35,37)$*	$(171,140,221)$*	$(252,115,277)$*
$(13,84,85)$*	$(14,48,50)$	$(189,180,261)$	$(270,144,306)$
$(15,112,113)$*	$(16,63,65)$*	$(207,224,305)$*	$(288,175,337)$*
$(17,144,145)$*	$(18,80,82)$	$(225,272,353)$*	$(306,208,370)$
$(19,180,191)$*	$(20,99,101)$*	$(243,324,405)$	$(324,243,405)$
$(21,220,221)$*	$(22,120,122)$	$(261,380,461)$*	$(342,280,442)$
$(23,264,265)$*	$(24,143,145)$*	$(279,440,521)$*	$(360,319,481)$*
$(25,312,313)$*	$(26,168,170)$	$(297,504,585)$	$(378,360,522)$

TABLE 1. The first 12 PT's for heights 1, 2, 81, and 162. The primitive PT's are starred.

As far as we can determine, the first use of the height and excess as parameters to enumerate PT's was by M. G. Teigan and D. W. Hadwin in [**8**]. The parameters used there are $x = h$, $y = \dfrac{e^2}{2h}$ (which, being $c - a$, is the height of (b, a, c)), and $z = e$. A similar method is found in [**6**]. The height-excess enumeration is implicit in [**9**], and explicit versions of it appear in [**2**] and [**10**].

We present a proof of the Height-Excess Enumeration Theorem in section 1. In section 2, we derive the Wade-Wade recursion formula, and in section 3, we briefly discuss another recursion formula for PT's, involving a generalized Fibonacci sequence, from the height-excess viewpoint.

1. Proof of the Height-Excess Enumeration Theorem

We first develop the key properties of d. As usual, the notation $x|y$ means that the integer y is divisible by the integer x.

Lemma. *Let h be a positive integer with associated increment d. Then $2h|d^2$. If D is any positive integer for which $2h|D^2$, then $d|D$.*

Proof. If p is odd, then $d^2 = 4p^2q^2 = 2p \cdot 2h$. If p is even, say $p = 2p_0$, then $d^2 = 4p_0^2q^2 = p_0 \cdot 2h$. So $2h|d^2$. For the second assertion of the Lemma, write D in terms of its distinct prime factors, $D = d_1^{r_1} \cdots d_k^{r_k}$, so that $D^2 = d_1^{2r_1} \cdots d_k^{2r_k}$. Similarly, write $p = p_1 \cdots p_m$ and $q^2 = q_1^{2t_1} \cdots q_n^{2t_n}$. When $2h|D^2$, each q_i must equal a d_j, with their exponents satisfying $2t_i \le 2r_j$ and hence $t_i \le r_j$. This shows that $q|D$, say $D = q \cdot D_1$ where $D_1 = s_1^{u_1} \cdots s_\ell^{u_\ell}$. Since $2pq^2|D^2$ and $D^2 = q^2D_1^2$, we have $2p|D_1^2$. So each p_i is one of the s_j, showing that $p|D_1$. Also, if p is odd, then since $2p|D_1^2$, 2 is one of the s_j, so $2p|D_1$. So $pq|D$, and if p is odd, then $2pq|D$. That is, $d|D$. □

Now we prove the Theorem. By the Lemma, the coordinates of $P(k,h)$ are positive integers, and using basic algebra one can verify that they satisfy the Pythagorean relation. Using $h = c-b$, $e = a+b-c$, and the Pythagorean relation, more algebra shows that for any PT,

$$(a,b,c) = \left(h+e, e+\frac{e^2}{2h}, h+e+\frac{e^2}{2h}\right) .$$

The Pythagorean relation implies that $e^2 = 2(c-a)(c-b)$, so $2h|e^2$. By the Lemma, e is divisible by d. So every PT equals some $P(k,h)$. Since any PT determines h and e uniquely, it also determines k uniquely, so it can equal only one $P(k,h)$.

We note that $P(k,h)$ is a triangle exactly when $h + dk < dk + \dfrac{(dk)^2}{2h}$, which says that $k > \dfrac{h}{d}\sqrt{2}$, so it remains only to establish the criterion for $P(k,h)$ to be primitive.

We will first show that if (a,b,c) is primitive, then $c-a$ and $c-b$ are relatively prime. If r were a prime dividing both of them, then r would divide the sum $(c-a)^2+(c-b)^2 = (3c-2a-2b)c$. Now r could not divide c, since then it would divide a and b and (a,b,c) would not be primitive. So r divides $3c-2a-2b = 2(c-a)+2(c-b)-c$, again giving the contradiction that r divides c. We conclude that $c-a = k^2\dfrac{d^2}{2h}$ and $c-b = h$ are relatively prime. For p odd, these are $2pk^2$ and pq^2, so $p = 1$, q is odd, and $\gcd(2k,q) = 1$. For p even, they are $k^2\dfrac{p}{2}$ and pq^2, so $p = 2$ and $\gcd(k,2q) = 1$.

Conversely, suppose that h and k satisfy the given conditions. For $h = q^2$, (a,b) is $(q(q+2k), 2k(q+k))$. If r is a prime dividing both entries, then $r \ne 2$ since the first entry is odd. So r must divide q or $q+2k$, and must divide k or $q+k$. Any of the four possible combinations leads to r dividing both q and k, a contradiction. For $h = 2q^2$, (a,b) is $(2q(k+q), k(2q+k))$ and the reasoning is similar.

2. The Wade-Wade recursion formula

Simple recursion formulas for generating PT's of a fixed height have long been known. An early example is [**1**], which provides recursions that start with a PT and produce some but not usually all of the other PT's with the same height. P. W. Wade and W. R. Wade [**9**] gave a recursion formula which generates all PT's of height h, verifying this fully for the cases when h is of the form q^2 or $2q^2$. Using the Height-Excess Enumeration Theorem, we will verify their recursion formula for all h. Putting $(a_k, b_k, c_k) = \left(h + dk, dk + \frac{(dk)^2}{2h}, h + dk + \frac{(dk)^2}{2h}\right)$, we have

$$\begin{aligned}
(a_{k+1}, b_{k+1}) &= \left(h + d(k+1), d(k+1) + \frac{(d(k+1))^2}{2h}\right) \\
&= \left(h + dk + d, dk + d + \frac{(dk)^2}{2h} + dk\,\frac{d}{h} + \frac{d^2}{2h}\right) \\
&= \left(a_k + d, \frac{d}{h}a_k + b_k + \frac{d^2}{2h}\right) .
\end{aligned}$$

so $(a_{k+1}, b_{k+1}, c_{k+1}) = (a_k + d, \frac{d}{h}a_k + b_k + \frac{d^2}{2h}, \frac{d}{h}a_k + c_k + \frac{d^2}{2h})$. If we start with $(a_0, b_0, c_0) = (h, 0, h)$, this recursion produces exactly the triples in the Theorem that have height h— first the finitely many PT's of height h that are not Pythagorean triangles, and then, once k exceeds $\frac{h}{d}\sqrt{2}$, the infinitely many Pythagorean triangles of height h. The triples in table 1 were calculated using the Wade-Wade recursion formula and *Mathematica.*

3. PT's and Fibonacci-like sequences

Other recursive methods for producing PT's have been developed. For example, the following method based on a generalized Fibonacci sequence was given by Horadam [**3, 4**]. Start with positive integers r and s. Put $H_1 = r$, $H_2 = s$, and $H_n = H_{n-1} + H_{n-2}$ for $n \geq 3$. The triples

$$(2H_{n+1}H_{n+2}, H_nH_{n+3}, 2H_{n+1}H_{n+2} + H_n^2)$$

are PT's with a even. Using the Fibonacci identity, this triple can be written as

$$(2H_{n+1}^2 + 2H_nH_{n+1}, H_n^2 + 2H_nH_{n+1}, H_n^2 + 2H_nH_{n+1} + 2H_{n+1}^2)$$

which is exactly $P(H_n, 2H_{n+1}^2)$.

Notice that in any primitive PT, c is odd and exactly one of a and b is odd, so a is even if and only if the height is even. So the primitive PT's with a even are exactly the $P(k, 2q^2)$ with k and $2q$ relatively prime. As noted in [**3**], these are Horadam PT's with $H_1 = k$ and $H_2 = q$.

Now, H_n and H_{n+1} are relatively prime if and only if r and s were relatively prime, so the Horadam PT is primitive exactly when r and s were relatively prime and H_n is odd. If r and s are relatively prime and H_n is even, then H_{n+1} is odd and $H_n/2$ and H_{n+1} are relatively prime. Thus, $P(H_n/2, H_{n+1}^2)$ is primitive, and one finds that $2P(H_n/2, H_{n+1}^2) = P(H_n, 2H_{n+1}^2)$, so the Horadam triple is 2 times a primitive PT (note that in general, however, $m\,P(k,h) \neq P(mk,mh)$, for example $P(1,2) = (4,3,5)$ but $P(2,4) = (12,16,20)$).

It would be interesting to see whether some of the more sophisticated Fibonacci recursion methods for producing triples, such as that of [**5**], can be analyzed using the height-excess enumeration.

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