Mathematics 2443-004 | Name (please print) ______________________
Examination I Form A | Student Number ______________
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I. (5) For the function $f(x, y) = x^3 - 3xy + y^3$, locate all critical points. For each critical point, use the second derivatives test ($D = f_{xx}f_{yy} - (f_{xy})^2$) to determine whether the critical point is a local maximum, local minimum, or neither.

II. (3) Let $f(x, y)$ be a function of two variables. Define what it means to say that (a, b) is a *critical point* of f.

III. (5) For the function $V(x, y, z) = e^{xyz}$, calculate $\nabla V(1, -1, -1)$. Use it to write an equation of the tangent plane to the level surface $e^{xyz} = e$ at the point $(1, -1, -1)$. (Recall that the equation of the plane with normal vector $a\vec{\imath} + b\vec{\jmath} + c\vec{k}$ through the point (x_0, y_0, z_0) is $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$.)

IV. (5) For the function $f(x, y) = \frac{x}{y}$, use the gradient vector to find the rate of change of f at the point $(2, 1)$ in the direction toward $(-1, 0)$. Find the maximum rate of change of f at $(2, 1)$, and the direction in which it occurs.

V. (5) At a certain moment, a piece of wire has the shape of a cylinder 100 mm long and 2 mm in diameter, and it is being stretched to that its length h increases at a rate of 0.5 mm/sec. Assuming that it remains cylindrical, and its volume does not change, how is its diameter changing at this moment? (Hint: Let V be the volume. One needs to find the value of $\frac{dr}{dt}$ that will make $\frac{dV}{dt} = 0$ at the time in question.)

VI. (5) Write out the chain rule for $\frac{\partial v}{\partial z}$ where $v = f(p,q,r)$, $p = p(x,y,z)$, $q = q(x,y,z)$, and $r = r(x,y,z)$, Use it to calculate $\frac{\partial v}{\partial z}$ in terms of p, q and r at the point $(x,y,z) = (1,2,4)$ if $f(p,q,r) = \cos(pqr)$, $\frac{\partial p}{\partial z} = 2x$, $\frac{\partial q}{\partial z} = -y$, and $\frac{\partial r}{\partial z} = xyz$.

VII. (6) Consider the problem of finding the maximum value of the function $x+y$ on the surface $4x^2+y^2 = 20$. Use Lagrange multipliers to set up a system of equations whose solutions include the point where the maximum value of $x+y$ occurs on the surface $4x^2+y^2 = 20$. Solve the equations to find where the maximum occurs. (Suggestion: unless you see quickly how to solve the equations, leave that part until you have worked out the other problems on the test.)

VIII. For each of the following limits, determine whether the limit exists, and if it exists, find its value:
(4)

$$\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}\ , \quad \lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+y^2}\ .$$

IX. **(Bonus problem)** Let $\gamma(t) = (x(t), y(t))$ be a parameterization of a level curve $f(x, y) = c$. By calculating
(4) $\dfrac{d(f(\gamma(t)))}{dt}$, verify that ∇f is perpendicular to the tangent vector to the level curve.