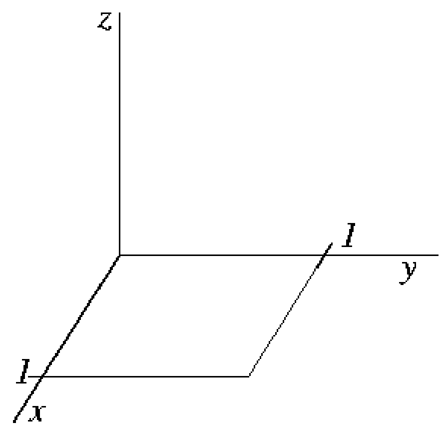
Mathematics 2443-004
Final Examination
May 5, 2000

Name (please print) ______________________

Student Number ______________

I. (10) (a) In the coordinate system shown below, draw a typical graph of a function $z = g(x, y)$ with domain the square $0 \le x \le 1$, $0 \le y \le 1$. Regard it as a parameterized surface with parameterization $x = x$, $y = y$, and $z = g(x, y)$.



(b) At some point on the graph, draw the vectors $\vec{r}_x$ and $\vec{r}_y$. Calculate explicit expressions for these vectors.

(c) Use the expressions in (b) to calculate $\vec{r}_x \times \vec{r}_y$.

(d) Calculate an expression for $\| \vec{r}_x \times \vec{r}_y \|$.

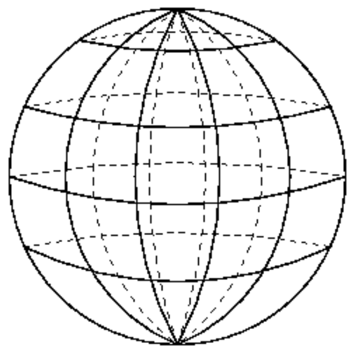
II. (10) (a) Use Stokes' Theorem $\iint_S \operatorname{curl} \vec{F} \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{R}$ to calculate $\int_C (x\vec{\imath} + x\vec{\jmath} + x\vec{k}) \cdot d\vec{r}$, where C is the unit circle in the xy-plane, oriented counterclockwise, and S is the unit disc in the xy-plane.

(b) Calculate the line integral in part (a) directly (i. e. using a parameterization $\vec{r}(t)$ of C).

III. (12) The equations $x = a\sin(\phi)\cos(\theta)$, $y = a\sin(\phi)\sin(\theta)$, $z = a\cos(\phi)$ give a parametric representation of the sphere $x^2 + y^2 + z^2 = a^2$.

1. Calculate $\vec{r}_\phi$ and $\vec{r}_\theta$.

2. On this picture of the sphere, label the lines where θ is constant, and those where ϕ is constant. Draw some of the vectors $\vec{r}_\phi$ and $\vec{r}_\theta$, and also some of the vectors $\vec{r}_\phi \times \vec{r}_\theta$.



3. Let S be the portion of the sphere with $z \geq 0$, that is, the top half of the sphere, and with the outward normal vector. Take as given the fact that $\vec{r}_\phi \times \vec{r}_\theta$ is $a^2 \sin(\phi)(x\vec{\imath} + y\vec{\jmath} + z\vec{k})$. Use the formula $\iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F} \cdot (\vec{r}_\phi \times \vec{r}_\theta)\, dD$ to calculate $\iint_S \vec{k} \cdot d\vec{S}$.

4. Use the Divergence Theorem to calculate $\iint_T \vec{k} \cdot d\vec{T}$, where T is the entire sphere $x^2 + y^2 + z^2 = a^2$, with the outward normal vector.

IV. Let $f(x, y, z)$ be a function of three variables. Suppose that a surface S is parameterized by the equations
(8) $x = x(u, v)$, $y = y(u, v)$, and $z = z(u, v)$.

1. Use the Chain Rule to get an expression for $\dfrac{\partial f}{\partial u}$.

2. Calculate $\vec{r}_u$.

3. Use ∇f to calculate $D_{\vec{r}_u} f$, the directional derivative of f in the direction of $\vec{r}_u$.

V. Use the Divergence Theorem $\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div} \vec{F}\, dV$ to calculate $\iint_S (\sin(x)\vec{\imath} + \sin(y)\vec{\jmath} + \sin(z)\vec{k}) \cdot d\vec{S}$,
(7) where S is the unit cube $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$ with the outward normal.

VI. Let R be a region in the plane. Define what it means to say that R is *simply-connected.* Explain how the
(4) definition shows that the region given in polar coordinates by $1 < r < 2$ is not simply-connected.

VII. (7) Let D be the unit disc consisting of all (x, y) for which $x^2 + y^2 \leq 1$. Let $f(x, y) = x^2 + xy^2$. Use the methods of section 12.7 (i. e. critical points and investigation of the values of f to the boundary of D) to determine the maximum and minimum values of f on D, and the point or points at which they occur.

VIII. Verify that the vector field $\cos(x^2)\sin(y^2)\,\vec{\imath} + \sin(x^2)\cos(y^2)\,\vec{\jmath}$ in the plane is not conservative.
(4)

IX. Calculate $\int_C (x^{99}y^{100}\vec{\imath} + x^{100}y^{99}\vec{\jmath}) \cdot d\vec{r}$, where C is the straight line from $(1, 1)$ to $(2\pi, 0)$.
(5)

X. Calculate the following partial derivatives.
(9)

1. f_v if $f(u,v) = \tan^{-1}(\frac{u}{v})$

2. $\dfrac{\partial R}{\partial R_2}$ if $\dfrac{1}{R} = \dfrac{1}{R_1} + \dfrac{1}{R_2} + \dfrac{1}{R_3}$

3. $\dfrac{\partial^2 w}{\partial r \partial \theta}$ if $w = \cos(r\theta)$