Mathematics 4433-001

Final Examination

May 3, 2000

Instructions: Give brief and to-the-point answers.

I. (13) (a) Give the ϵ-δ definition of $\lim_{x\to x_0} f(x) = L$.

(b) Use the ϵ-δ definition to prove that $\lim_{x\to 1} 1 + x = 2$.

(c) Use the ϵ-δ definition to prove that $\lim_{x\to 1}(1 + x)^2 = 4$.

(d) Give the precise definition of $\lim_{x\to\infty} f(x) = L$.

(e) Give the precise definition of $\lim_{x\to x_0} f(x) = \infty$.

II. (4) Define $f\colon \mathbb{R} \to \mathbb{R}$ by $f(x) = \sin(1/x)$ if $x \neq 0$ and $f(0) = 0$. Use proof by contradiction and the ϵ-δ definition of continuity to prove that f is not continuous at $x_0 = 0$.

III. (4) Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ are continuous functions. Use the ϵ-δ definition of continuity to prove that $f + g$ is continuous.

IV. (5) (a) Define what it means to say that a function $f\colon E \to \mathbb{R}$ is *uniformly continuous.*

(b) Use the Mean Value Theorem to prove that if $f\colon \mathbb{R} \to \mathbb{R}$ is a differentiable function and there is a number M so that $|f'(x)| \leq M$ for all $x \in \mathbb{R}$, then f is uniformly continuous.

V. (3) Write the ϵ-δ definition of the statement that f is *not* continuous at x_0.

VI. (4) Let $f\colon [0, 1] \to [0, 1]$ be continuous. Prove that there is a number $c \in [0, 1]$ such that $f(c) = c$.

VII. (10) Without verifying details, give examples of the following.

(a) A function $f\colon \mathbb{R} \to \mathbb{R}$ which is continuous at every irrational number and discontinuous at every rational number.

(b) A sequence f_n of Riemann integrable functions on $[0, 1]$ that converges to a Riemann integrable function f, but the sequence of real numbers $\int_a^b f_n$ does not converge to $\int_a^b f$.

(c) A sequence of Riemann integrable functions on $[0, 1]$ that converges to a function which is not Riemann integrable.

(d) A sequence of functions on $[0, 1]$, each of which is continuous at all $x \neq 1/2$ but is discontinuous at $x = 1/2$, that converges uniformly to a continuous function.

(e) A bounded function $f\colon (0, 1) \to \mathbb{R}$ which is continuous but not uniformly continuous.

VIII. (4) Give an example of a function $f\colon \mathbb{R} \to \mathbb{R}$ which is continuous but not differentiable at every $x \in \mathbb{R}$. (First, define f_1 by giving its graph, then define functions f_n, and use a series to define f.)

IX. (4) Let $f\colon [2, 4] \to \mathbb{R}$ be a continuous function. Suppose that $f(2) = 2$, $f(3) = 4$, and $f(4) = 3$. Prove that f is not monotone.

X. (10) (a) Define what it means to say that a set E of real numbers *has measure* 0.

(b) Prove that the set $E = \{1/n \mid n = 1, 2, 3, \dots\}$ has measure 0.

(c) State the Riemann-Lebesgue theorem.

(d) Use the Riemann-Lebesgue theorem to prove that if $f\colon [a,b] \to \mathbb{R}$ and $g\colon [a,b] \to \mathbb{R}$ are Riemann integrable, then their sum $f + g\colon [a,b] \to \mathbb{R}$ is Riemann integrable.

XI. (7) Let $\sum_{j=1}^{\infty} f_j$ be a series of functions, each with domain E.

(a) Define what it means to say that $\sum_{j=1}^{\infty} f_j$ *converges* to $f\colon E \to \mathbb{R}$.

(b) Define what it means to say that $\sum_{j=1}^{\infty} f_j$ converges *uniformly* to $f\colon E \to \mathbb{R}$.

(c) Suppose that a power series $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence $R = 5$. What can be said about the convergence behavior of $\sum_{n=0}^{\infty} a_n x^n$, and about where it is uniform?

XII. (4) Let A and B be nonempty sets of real numbers which have upper bounds. Suppose that $\forall a \in A, \exists b \in B, a \le b$. Prove that $\sup(A) \le \sup(B)$.

XIII. (4) Use the Weierstrass M-test to verify that the series $\sum_{n=0}^{\infty} x^n$ converges uniformly on the interval $[-1/2, 1/2]$.

XIV. (4) Let $f\colon [a,b] \to \mathbb{R}$ and $g\colon [a,b] \to \mathbb{R}$ be bounded functions, and let P be a partition of $[a,b]$. Prove the following fact, which is a key step in proving that $\int_a^b f + g = \int_a^b f + \int_a^b g$: $m_i(f+g) \ge m_i(f) + m_i(g)$.