Mathematics 4433-001

Examination III

April 21, 2000

Instructions: Give brief and to-the-point answers.

I. Let $f_n\colon E \to \mathbb{R}$ be a sequence of functions, and let $f\colon E \to \mathbb{R}$ be a function.
(12)

(a) Define what it means to say that the sequence f_n converges *uniformly* to f.

(b) Let $M_n = \sup_{x\in E} |f_n(x) - f(x)|$. Prove that if the f_n converge uniformly, then $\lim M_n = 0$.

(c) State the Cauchy Criterion for Uniform Convergence of a Sequence of Functions.

II. (a) State the Mean Value Theorem for Integrals.
(11)

(b) Show that the Mean Value Theorem for Integrals need not hold if the function is not continuous.

(c) What major theorem is used in the proof of the Mean Value Theorem for Integrals?

III. Take as given the fact that if $g\colon [a,b] \to \mathbb{R}$ is a Riemann integrable function that satisfies $g(x) \geq 0$ for all
(8) $x \in [a,b]$, then $\int_a^b g \geq 0$. Let $f\colon [a,b] \to \mathbb{R}$ be a continuous function, and suppose that $\int_a^b f^2 = 0$. Let $F\colon [a,b] \to \mathbb{R}$ be the function defined by $F(x) = \int_a^x f^2$.

(a) Verify that $F(x) = 0$ for all $x \in [a,b]$.

(b) Deduce that $f(x) = 0$ for all $x \in [a,b]$.

IV. Take as given the following fact: If $f\colon [a,b] \to \mathbb{R}$ is a Riemann integrable function, and $g\colon [a,b] \to \mathbb{R}$ is
(8) a function with $g(x) = f(x)$ if $x \neq c$, then g is Riemann integrable and $\int_a^b g = \int_a^b f$. Prove the following fact: If $f\colon [a,b] \to \mathbb{R}$ is a Riemann integrable function, and $g\colon [a,b] \to \mathbb{R}$ is a function with $g(x) = f(x)$ if $x \notin \{c_1, c_2, \ldots, c_n\}$, then g is Riemann integrable and $\int_a^b g = \int_a^b f$.

V. Without verifying details, give examples of the following:
(12)

(a) A sequence f_n of Riemann integrable functions that converges to a Riemann integrable function f, but the sequence of real numbers $\int_a^b f_n$ does not converge to $\int_a^b f$.

(b) A sequence of functions that converges uniformly on $[0,M]$ for each $M > 0$, but does not converge uniformly on $[0,\infty)$.

(c) A sequence of functions f_n that converges uniformly on $[-1,1]$, but whose derivatives at zero $f_n'(0)$ do not converge.

(d) A sequence of continuous functions on $[0,1]$ that converges to a continuous function pointwise, but not uniformly.

VI. Let $f\colon [a,b] \to \mathbb{R}$ and $g\colon [a,b] \to \mathbb{R}$ be bounded functions, and let P be a partition of $[a,b]$. Prove the
(6) following fact, which was a key step in proving that $\int_a^b f + g = \int_a^b f + \int_a^b g$: $M_i(f+g) \leq M_i(f) + M_i(g)$.