Mathematics 3113-002 Name (please print) ________________

Final Examination

May 9, 2001

I. (8) Calculate the *inverse* Laplace transforms of the following functions of s, following any special instructions given.

1. $\dfrac{e^{-2s}}{s^2+5} + \dfrac{e^{-2s}s}{s^2+5}$

2. $\dfrac{1}{s(s^2+1)}$, using the formula involving $\dfrac{1}{s}\mathcal{L}(f(t))$

II. (6) Give an implicit solution for $\dfrac{dy}{dx} = \dfrac{1+\sqrt{x}}{1+\sqrt{y}}$, $y(1) = 3$.

III. (8) Did you ever wonder what would happen if the method of variation of parameters were applied to a *first order* linear equation? Let's figure it out. Start with a general first-order linear equation $y' + p(x)y = q(x)$. Suppose that $y_1(x)$ is some nonzero solution of the associated homogeneous equation $y' + p(x)y = 0$. Vary it to get a trial solution for the nonhomogeneous equation, that is, write $y(x) = u_1(x)y_1(x)$.

1. Calculate $y'(x)$.

2. Put the expressions for $y(x)$ and $y'(x)$ into the nonhomogeneous equation $y' + p(x)y = q(x)$. Simplify it using the fact that $y_1(x)$ is a solution to the associated homogeneous equation, and solve for $u_1'(x)$. Use this expression to obtain a formula giving $y(x)$ as an integral whose integrand involves $q(x)$.

3. Now, let's see why this is nothing new. Use an integrating factor to find a solution $y_1(x)$ for $y' + p(x)y = 0$ (write $A(x)$ for an antiderivative of $p(x)$, multiply through by $e^{A(x)}$, and solve for y using the fact that $\int 0\,dx = C$).

4. Put the expression for $y_1(x)$ into the formula you obtained using variation of parameters and simplify. Not surprisingly, the resulting formula is exactly the one that results when one solves $y' + p(x)y = q(x)$ using an integrating factor, although you need not check this.

IV. Use the series method to solve the first order linear equation $y' + ky = 0$, $y(0) = a_0$ as follows.
(9)

1. Write $y = \sum_{n=0}^{\infty} a_n x^n$. Calculate y', and put the series expresssions for y' and y into the equation. Obtain a formula for a_{n+1} in terms of a_n.

2. Calculate a_1 in terms of a_0, a_2 in terms of a_0, and so on, until you see the general formula for a_n in terms of a_0.

3. Put the expressions for a_n in terms of a_0 into the series $\sum_{n=0}^{\infty} a_n x^n$. Simplify, using the Maclaurin series $e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!}$, to obtain the solution to the original equation.

V. (6) Consider the following system of differential equations

$$\begin{aligned} x' + 4x &= y' + y - t^3 \\ 3x' - y' &= 1 \end{aligned}$$

1. Rewrite the system using differential operator notation.

2. Use Kramer's rule to write a linear differential equation whose solution is x, but do *not* try to solve for x and y.

VI. (6) Consider the following system of differential equations

$$\begin{aligned} x' + 4x &= y' + y - t^3 \\ 3x' - y' &= 1 \end{aligned}$$

1. Assuming that $x(0) = 1$ and $y(0) = 0$, use the Laplace transform to rewrite the system in terms of functions of s that do not involve derivatives.

2. Use Kramer's rule to solve for $X(s)$, but do *not* try to find its inverse transform.

VII. Calculate the Laplace transforms of the following functions of t.
(8)
1. $t\, e^{4t} \sinh(t)$

2. $f(t) = t - [[\, t\,]]$, where $[[\, t\,]]$ means the greatest integer less than t (so $[[\, t\,]] = 0$ when $0 \le t < 1$, $[[\, t\,]] = 1$ when $1 \le t < 2$, and so on). You might need to carry out an integration, although it can be avoided by clever use of step functions.

VIII. Consider the boundary value problem $y'' + \lambda\, y = 0$; $y(0) = 0$, $y'(1) = 0$.
(9)

1. Define what it means to say that a number λ_i is an *eigenvalue* for the boundary value problem. Define what it means to say that a function is an *eigenfunction associated to* λ_i.

2. Show that this boundary value problem has no negative eigenvalues (start by writing $\lambda = -\alpha^2$, with $\alpha > 0$).

3. Find all positive eigenvalues of this boundary value problem, and associated eigenfunctions (start by writing $\lambda = \alpha^2$, with $\alpha > 0$).

IX. (8) Two functions $y = y_1(x)$ and $y = y_2(x)$ are linearly independent solutions to a certain linear differential equation $y'' + p(x)y' + q(x)y = 0$.

1. Write a system of linear equations to find c_1 and c_2 so that the solution $y(x) = c_1y_1(x) + c_2y_2(x)$ satisfies the initial conditions $y(0) = 3$ and $y'(0) = 8$. Use Kramer's rule to calculate c_2 in terms of $y_1(0)$, $y_1'(0)$, $y_2(0)$, and $y_2'(0)$.

2. Given that the function $y_3(x)$ satisfies the differential equation $y'' + p(x)y' + q(x)y = f(x)$, write a general solution of the differential equation $y'' + p(x)y' + q(x)y = f(x)$ in terms of $y_1(x)$, $y_2(x)$, and $y_3(x)$.

X. (7) For the equation $y^{(3)} + y' = 1 + x\cos(x)$, solve the associated homogeneous equation, then use the method of undetermined coefficients to write a trial solution for the equation, but do *not* try to proceed further with finding the solutions. The general formula for the method of undetermined coefficients is $x^s\big((A_0 + A_1x + \cdots + A_mx^m)e^{rx}\cos(kx) + (B_0 + B_1x + \cdots + B_mx^m)e^{rx}\sin(kx)\big)$.

(End of exam)