Mathematics 3113-002 Name (please print) ____________________

Examination II

March 14, 2001

I. The functions $y_1 = \cos(2x)$ and $y_2 = \sin(2x)$ are solutions to the differential equations $y'' + 4y = 0$.
(13)

1. Calculate the Wronskian $W(\cos(2x), \sin(2x))$.

2. Find a solution y of the differential equation $y'' + 4y = 0$ that satisfies the initial conditions $y(0) = 3$ and $y'(0) = 8$.

3. Given that the function $\dfrac{e^x}{5}$ satisfies the differential equation $y'' + 4y = e^x$, write a general solutions of the differential equation $y'' + 4y = e^x$.

II. (10) Using the formula $x^s\big((A_0 + A_1x + \cdots + A_mx^m)e^{rx}\cos(kx) + (B_0 + B_1x + \cdots + B_mx^m)e^{rx}\sin(kx)\big)$, write trial solutions for the method of undetermined coefficients for the following differential equations, but *do not* substitute them into the equations or proceed further with finding the solution.

1. $y'' + y = x\cos(x)$

2. $y^{(3)} + 3y'' + 3y' + y = xe^{-x}$ (Fact: $\lambda^3 + 3\lambda^2 + 3\lambda + 1 = (\lambda + 1)^3$)

III. (6) Rewrite $2\cos(7x) - 11\sin(7x)$ in phase-angle form. Give the exact function (so your answer will involve the inverse tangent function).

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IV. (5) Verify that the functions $(x+1)^2$, x^2, x, and 1 are linearly dependent on the interval of all real numbers.

V. (10) Making use of the equations $u_1'y_1+u_2'y_2=0$, $u_1'y_1'+u_2'y_2'=f(x)$, and the integration formula $\int \sin^2(x)\,dx = \frac{1}{2}x - \frac{1}{2}\sin(x)\cos(x)$, and the fact that $y = c_1\cos(x) + c_2\sin(x)$ is a general solution of the homogeneous differential equation $y'' + y = 0$, apply the method of variation of parameters to find a particular solution of $y'' + y = \sin(x)$. (Hint: You will know if you are on the right track if you find that $u_2' = \sin(x)\cos(x)$. Then, $u_2 = \int \sin(x)\cos(x)\,dx = \sin^2(x)/2$.)

VI. (14) Consider the boundary value problem $y'' + \lambda y = 0$; $y'(0) = 0$, $y(1) = 0$.

1. Define what it means to say that a number λ_i is an *eigenvalue* for the boundary value problem. Define what it means to say that a function is an *eigenfunction associated to* λ_i.

2. Complete the following argument which shows that this boundary value problem has no negative eigenvalues.

   Write $\lambda = -\alpha^2$ with $\alpha > 0$. The characteristic equation is $r^2 - \alpha^2 = 0$, with roots $\pm\alpha$, so the general solution is $y = c_1 e^{\alpha x} + c_2 e^{-\alpha x}$. We calculate that $y' = \alpha c_1 e^{\alpha x} - \alpha c_2 e^{-\alpha x}$. ...

3. Complete the following argument to find all positive eigenvalues of this boundary value problem, and associated eigenfunctions.

   Write $\lambda = \alpha^2$ with $\alpha > 0$. The characteristic equation is $r^2 + \alpha^2 = 0$, with roots $\pm\alpha\, i$, so the general solution is $y = c_1 \cos(\alpha x) + c_2 \sin(\alpha x)$. We calculate that $y' = -\alpha c_1 \sin(\alpha x) + \alpha c_2 \cos(\alpha x)$. ...