Mathematics 2443-003 Name (please print) ____________________

Final Examination Form A

May 7, 2003

I. (12) Let C be the circle $x^2 + y^2 = 1$ in the xy-plane, oriented counterclockwise. Regard C as parameterized by the vector-valued function $\vec{r}(t) = \cos(t)\vec{\imath} + \sin(t)\vec{\jmath}$, $0 \le t \le 2\pi$. Note that the velocity vector $\vec{r}\,'(t) = -\sin(t)\vec{\imath} + \cos(t)\vec{\jmath} = -y\vec{\imath} + x\vec{\jmath}$ has length 1, so it equals the unit tangent vector $\vec{T}$.

1. Calculate $\int_C (y\vec{\imath} - x\vec{\jmath}) \cdot d\vec{r}$, directly from the definition of the line integral of a vector field.

2. Calculate $\int_C (y\vec{\imath} - x\vec{\jmath}) \cdot d\vec{r}$, by direct calculation using the fact that $\int_C (P\vec{\imath} + Q\vec{\jmath}) \cdot d\vec{r} = \int_C P\,dx + Q\,dy$.

3. Calculate $\int_C (y\vec{\imath} - x\vec{\jmath}) \cdot d\vec{r}$, using the interpretation of the line integral of a vector field as an integral involving the expression $\vec{F} \cdot \vec{T}$.

4. Calculate $\int_C (y\vec{\imath} - x\vec{\jmath}) \cdot d\vec{r}$, using Green's Theorem.

II. (3) Tell what Clairaut's Theorem says.

III. (9) Let S be the surface given by $x = u\cos(v)$, $y = u\sin(v)$, and $z = u$, where the domain of the parameterization is the rectangle $0 \le u \le 1$ and $0 \le v \le 2\pi$.

1. Carry out the steps in the computation that $\| \vec{r}_u \times \vec{r}_v \| = \sqrt{2}\,u$.

2. Calculate $\iint_S x^2\,dS$.

3. Calculate $\iint_S z^2\vec{k} \cdot d\vec{S}$.

IV. (8) State Stokes' Theorem (not all the hypotheses, just the main formula). Use it to calculate the line integral $\int_C (x\vec{\imath} + y\vec{\jmath} + (y^2 - x^2)\,\vec{k}\,) \cdot d\vec{r}$, where C is the boundary of the part of the paraboloid $z = 1 - x^2 - y^2$ that lies in the first octant (assume that C is oriented counterclockwise when viewed from above).

V. (8) Verify that the Divergence Theorem is true for the vector field $\vec{F}(x, y, z) = x\vec{\imath} + y\vec{\jmath} + z\vec{k}$ on the region E which is the unit ball $x^2 + y^2 + z^2 \leq 1$.

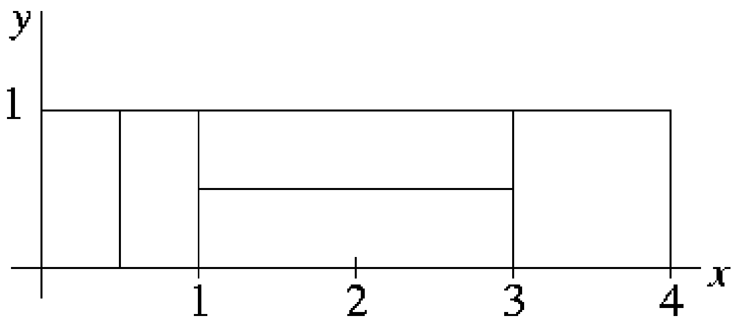
VI. Use the Fundamental Theorem of Calculus to carry out a partial calculation of $\iint_R \frac{\partial P}{\partial x}\, dA$, where R is the
(3) rectangle $-1 \le x \le 1$, $0 \le y \le 2$, and $P(x, y)$ is a function of x and y.

VII. Find a vector field $\vec{F}$ in the plane so that if C is any path in the plane, and C starts at P and ends at
(3) Q, then $\int_C \vec{F} \cdot d\vec{r}$ equals the product of the x- and y-coordinates of Q, minus the product of the x- and y-coordinates of P.

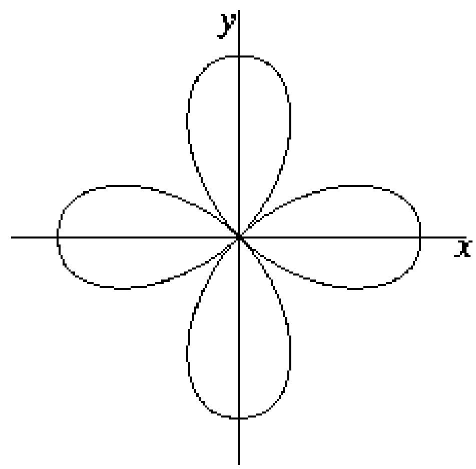
VIII. Use the gradient to find the directional derivative of $f(x, y) = \sin(xy)$ at the point $(2, 1)$ in the direction
(4) toward $(3, -1)$.

IX. Use implicit differentiation to calculate $\dfrac{\partial R}{\partial R_1}$ if $\dfrac{1}{R^3} = \dfrac{1}{R_1^2} + \dfrac{1}{R_2^2} + \dfrac{1}{R_3^2}$.
(3)

X. (4) Give the Riemann sum that approximates the integral $\int_0^1 \int_0^4 xy\ dx\,dy$ using the partition of the domain shown to the right, and taking the function values at the midpoints of the rectangles. Leave the answer as a sum of fractions, do not compute their sum.



XI. (4) The figure to the right shows the graph of the polar equation $r = \cos(2\theta)$. Use a double integral in polar coordinates to calculate the area contained inside each one of its loops.



XII. (6) Let E be the portion of the unit ball that has its x and y-coordinates positive, that is, the region $x^2+y^2+z^2 \leq 1$, $x \geq 0$, $y \geq 0$. Let S be the boundary surface of E, with the outward normal. Use the Divergence Theorem to calculate $\iint_S (x^3\vec{\imath} + y^3\vec{\jmath} + z^3\vec{k}) \cdot d\vec{S}$.

XIII. For the function $f(x, y) = \sqrt{x^2 + y^2} - \frac{1}{2}\ln(x^2 + y^2)$, find all critical points of f.
(4)

XIV. A certain function $f(x, y)$ has gradient $e^{x^2}\,\vec{\imath} + y\cos(y^2)\vec{\jmath}$. Its value at $(0, 0)$ is 6. Find its value at $(0, 1)$.
(4) (Note: you cannot integrate e^{x^2}, so you cannot calculate an explicit expression for $f(x, y)$).

XV. Let C be the portion of the unit circle that lies in the first quadrant. Find the maximum value of the
(4) function $f(x, y) = x^2 y$ on C by parameterizing C as $x = \cos(\theta)$, $y = \sin(\theta)$ and regarding f as a function of θ.