Mathematics 2423-001H Name (please print) ____________________

Final Examination

May 11, 2005

Instructions: Remember that even if you cannot do one part of a problem, you may assume that it is true and use it to do later parts of the problem.

**I.** Calculate the following integrals using integration by parts.
(8)

1. $\displaystyle\int xe^x\,dx$
2. $\displaystyle\int \frac{x^3}{\sqrt{1+x^2}}\,dx$

**II.** Let $\tan^{-1}(x)$ be the inverse of the function $f(x) = \tan(x)$, $-\pi/2 < x < \pi/2$.
(10)

1. Find the domain and range of $\tan^{-1}(x)$.
2. Sketch the graph of $\tan^{-1}(x)$.
3. Use right triangles to simplify the expressions $\csc(\tan^{-1}(x))$ and $\cos\left(2\tan^{-1}\left(\frac{\sqrt{x}}{2}\right)\right)$.
4. Use the chain rule to calculate the derivative of $\tan^{-1}(x)$, and write the corresponding indefinite integral formula.

**III.** We know, of course, that the exact value of $\displaystyle\int_0^\pi \sin(x)\,dx$ is 2. Calculate the value obtained when Simpson's Rule with $n = 4$ is used to estimate $\displaystyle\int_0^\pi \sin(x)\,dx$. (Find the exact value of the estimate; its numerical value is approximately 2.00455.) Use one of the error formulas to estimate the error. (Leave the error estimate as an expression involving $\pi$; in case you are curious, its numerical value is close to 0.00664, so it gives a rather accurate estimate of the error.)
(8)

**IV.** Consider the portion of the graph $y = \tan^{-1}(x)$ between $x = 0$ and $x = 1$. For each of the following, write an integral whose value is the specified quantity for this portion of the graph, but do not attempt to evaluate the integrals.
(8)

1. The length of this portion of the graph.
2. The surface area obtained when it is rotated about the $x$-axis.
3. The surface area obtained when it is rotated about the line $y = -1$.
4. The surface area obtained when it is rotated about the $y$-axis.

**V.** Use a trig substitution to evaluate the integral $\displaystyle\int \sqrt{1+4x^2}\,dx$. You may want to utilize the table of integrals for some of the later steps in the calculation. Express the answer in terms of $x$.
(8)

**VI.** (9) This problem concerns functions that are one-to-one.

1. Give a formal definition (not just the intuitive idea) of the statement that a function $f$ is *one-to-one*.
2. Give an example of two functions (defined for all $x$) that are one-to-one, but whose product is not one-to-one.
3. Give an example of two functions (defined for all $x$) that are not one-to-one, but whose product is one-to-one.

**VII.** (8) Use the table of integrals to calculate $\int \frac{1}{x^2(3x-1)}\,dx$. Calculate $\int_1^\infty \frac{1}{x^2(3x-1)}\,dx$.

**VIII.** (3) Give an explicit example of a partition of the interval $[0, e]$ that has mesh $10^{-2}$. You may use the fact that $e$ is approximately 2.718281828459

**IX.** (5) Calculate the Riemann sum for the following partition and function, using right-hand endpoints as the sample points $x_i^*$: the function is $f(x) = x$, the interval is $[0, 2]$, and the partition has $n = 4$ with $x_1 = 0.89$, $x_2 = 1.89$, and $x_3 = 1.99$. Leave the answer as a sum of expressions involving decimal numbers; do not carry out the arithmetic.

**X.** (5) A continuous function $f(x)$ is positive and *increasing* for $0 \le x \le 1$. A partition $0 = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = 1$ is selected. Let $L = \sum_{i=1}^n f(x_{i-1})\,\Delta x_i$ be the Riemann sum for $f(x)$ computed using *left-hand endpoints* as the sample points, and let $R = \sum_{i=1}^n f(x_i)\,\Delta x_i$ be the Riemann sum for $f(x)$ computed using *right-hand endpoints* as the sample points. Using pictures to clarify your explanation, and regarding $\int_0^1 f(x)\,dx$ as the area under $y = f(x)$ between $x = 0$ and $x = 1$, explain why $L < \int_0^1 f(x)\,dx$ an $\int_0^1 f(x)\,dx < R$.

**XI.** (4) State the Fundamental Theorem of Calculus (both parts, of course).

**XII.** (4) Verify that $y = a\sinh(x) + b\cosh(x)$ is a solution to the differential equation $y'' - y = 0$.

**XIII.** (8) Use l'Hôpital's rule to calculate the following limits:

1. $\lim_{t\to 0} \frac{5^t - 3^t}{t}$.
2. $\lim_{x\to\infty} \left(1 + \frac{\pi}{x}\right)^x$.

**XIV.** (6) For each of the following rational functions, write out the form of the partial fraction decomposition. Do not solve for unknown values of the coefficients.

1. $\frac{x^4 - x^2}{(x^2+1)^3}$
2. $\frac{1}{(x+1)^2(x+3)^2(x^2-1)^2}$

**XV**. Calculate the derivatives of the following functions:
(6)

1. $\displaystyle\int_2^{\ln(x)} \frac{1}{\ln(t)}\, dt.$

2. $\displaystyle\int_0^{\int_0^{x^2} e^{u^2}\, du} e^{t^2}\, dt.$

**XVI**. Let $R$ be the region between $y = 0$ and $y = \frac{1}{x}$ for $1 \leq x < \infty$.
(12)

1. Calculate the volume of the solid $E$ obtained when $R$ is rotated about the $x$-axis (the volume is given by an improper integral, whose value you will need to calculate).

2. Write an improper integral whose value represents the surface area of $E$ (not including the side disk where $x = 1$, just the part produced by rotating $y = \frac{1}{x}$).

3. By making a comparison, verify that the integral which represents the surface area of $E$ diverges to $\infty$.

**XVII**. Use integration by parts to verify that $f(a+h) - f(a) - f'(a)h = \displaystyle\int_0^h (h-t)f''(a+t)\, dt.$
(6)

Table of Integrals

$$\int \sec(u)\,du = \ln|\sec(u)+\tan(u)| + C,$$

$$\int \csc(u)\,du = \ln|\csc(u)-\cot(u)| + C$$

$$\int \sec^n(u)\,du = \frac{1}{n-1}\tan(u)\sec^{n-2}(u) + \frac{n-2}{n-1}\int \sec^{n-2}(u)$$

$$\int \csc^n(u)\,du = \frac{-1}{n-1}\cot(u)\csc^{n-2}(u) + \frac{n-2}{n-1}\int \csc^{n-2}(u)$$

$$\int \frac{u^2}{\sqrt{u^2-a^2}}\,du = \frac{u}{2}\sqrt{u^2-a^2} + \frac{a^2}{2}\ln|u+\sqrt{u^2-a^2}| + C$$

$$\int \frac{u^2}{\sqrt{2au-u^2}}\,du = -\frac{u+3a}{2}\sqrt{2au-u^2} + \frac{3a^2}{2}\cos^{-1}\left(\frac{a-u}{a}\right) + C$$

$$\int \frac{u^2}{\sqrt{a^2-u^2}}\,du = -\frac{u}{2}\sqrt{a^2-u^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{u}{a}\right) + C$$

$$\int \frac{u^2}{\sqrt{a+bu}}\,du = \frac{2}{15b^3}(8a^2+3b^2u^2-4abu)\sqrt{a+bu} + C$$

$$\int \frac{1}{\sqrt{a^2+u^2}}\,du = \ln|u+\sqrt{u^2+a^2}| + C$$

$$\int \frac{du}{u(a+bu)} = \frac{1}{a}\ln\left|\frac{u}{a+bu}\right| + C$$

$$\int \frac{du}{u^2(a+bu)} = -\frac{1}{au} + \frac{b}{a^2}\ln\left|\frac{a+bu}{u}\right| + C$$

$$\int \frac{u}{(a+bu)^2}\,du = \frac{a}{b^2(a+bu)} + \frac{1}{b^2}\ln|a+bu| + C$$

$$\int \frac{du}{u(a+bu)^2} = \frac{1}{a(a+bu)} - \frac{1}{a^2}\ln\left|\frac{a+bu}{u}\right| + C$$

$$\int \frac{1}{a^2+u^2}\,du = \frac{1}{a}\arctan\left(\frac{u}{a}\right) + C$$

Simpson's Rule: $\int_a^b f(x)\,dx \approx \frac{h}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + \cdots + 4y_{n-1} + y_n)$ with error of magnitude at most $\frac{K(b-a)^5}{180n^4} = \frac{K(b-a)}{180}h^4$, where $|f^{(4)}(x)| \le K$ for $a \le x \le b$.