Mathematics 2443-006H Name (please print) ______________________

Final Examination

May 11, 2006

Instructions: Give brief, clear answers. Use theorems whenever possible.

I. (9) Let S be the portion of the cone $z = \sqrt{x^2+y^2}$ between $z = 0$ and $z = 2$. It can be parameterized by the formulas $x = u\cos(v)$, $y = u\sin(v)$, $z = u$.
(a) Sketch the domain R of this parameterization.
(b) Sketch the surface and some typical vectors $\vec{r}_u$ and $\vec{r}_v$.
(c) Calculate $\vec{r}_u$ and $\vec{r}_v$ explicitly, and use them to calculate an upward normal vector to the surface.
(d) Express dS in terms of dR.

II. (5) It is a fact that any simple closed loop C in 3-dimensional space bounds a two-sided surface S (although if C is knotted, S will not be a disk, but a more complicated surface), and Stokes' Theorem applies to any surface bounded by C, not just disks. Using this fact, together with Stokes' Theorem, verify that if C is any simple closed loop, $\int_C \nabla f\cdot d\vec{r} = 0$. (Of course, this follows from the Fundamental Theorem for Line Integrals as well.) Verify any facts about curl that may be needed in your argument.

III. (6) Verify that if S and E satisfy the hypotheses of the Divergence Theorem, then:
(a) the volume of E is $\frac{1}{3}\iint_S (x\vec{\imath}+y\vec{\jmath}+z\vec{k})\cdot d\vec{S}$.
(b) $\iint_S D_{\vec{n}}f\ dS = \iiint_E \Delta f\, dV$, where $\vec{n}$ is the unit normal to the surfaces and Δf is the Laplacian $f_{xx}\vec{\imath}+f_{yy}\vec{\jmath}+f_{zz}\vec{k}$.

IV. (6) Use the Divergence Theorem to calculate $\iint_S (4x^3z\vec{\imath}+4y^3z\vec{\jmath}+3z^4\vec{k})\cdot d\vec{S}$ where S is the boundary of the solid hemisphere $x^2+y^2+z^2\le R^2$, $0\le z$. Hint: use spherical coordinates on the solid hemisphere, and the fact that $x^2+y^2+z^2=\rho^2$ to simplify the integrand.

V. (6) Use Stokes' Theorem to evaluate $\int_C (e^{-x}\vec{\imath}+e^{x}\vec{\jmath}+e^{z}\vec{k})\cdot d\vec{r}$, where C is the boundary of the portion of the surface $x+y+z=1$ that lies in the first octant. You may take as known the fact that $\mathrm{curl}(e^{-x}\vec{\imath}+e^{x}\vec{\jmath}+e^{z}\vec{k}) = e^x\vec{k}$, no need to calculate it.

VI. (5) Use implicit differentiation to calculate $\left.\dfrac{\partial R}{\partial R_3}\right|_{(R_1,R_2,R_3)=(\sqrt{3},\sqrt{6},2)}$ if $\dfrac{1}{R} = \dfrac{1}{R_1^2}+\dfrac{1}{R_2^2}+\dfrac{1}{R_3^2}$.

VII. (6) If z is a function of x and y, calculate $\dfrac{\partial z}{\partial r}$ and $\dfrac{\partial z}{\partial \theta}$, where r and θ are the polar coordinates. Write each result in terms of $\dfrac{\partial z}{\partial x}$, $\dfrac{\partial z}{\partial y}$, x, y, and r, that is, without using θ explicitly.

VIII. (12) Calculate each of the following.
(a) The directional derivative of $\dfrac{1}{xy}+\dfrac{1}{yz}$ at $(2,1,2)$ in the direction toward the origin.
(b) The maximum rate of change of $qe^{-p}-pe^{-q}$ at $(p,q)=(0,0)$, and the direction in which it occurs.
(c) A vector-valued function giving the line perpendicular to the level surface of xyz at the point $(1,2,3)$.
(d) An equation for the tangent plane to the level surface of $\dfrac{1}{xyz}$ at the point $(1,2,3)$.

IX. (5) Six positive numbers x, y, z, u, v, and w, each less than or equal to 2, are multiplied together. Use differentials to estimate the maximum possible error in the computed product that might result from rounding each number off to the nearest whole number.

X. (5) Six positive numbers x, y, z, u, v, and w, are multiplied together. The first three are increasing at 0.5 units per second, while the last three are decreasing at 0.1 units per second. Find the rate of change of the product at a moment when all of the numbers except w equal 1, and $w = 2$.

XI. (7) Calculate the area inside the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 3$ as follows.

(a) Let S be the region bounded by the ellipse. Define ϕ from the uv-plane to the xy-plane by $\phi(u,v) = (au, bv)$. Determine the region R in the uv-plane that corresponds to S under ϕ.

(b) Calculate the Jacobian $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{pmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} \end{pmatrix}$ and its determinant.

(c) Write a double integral over the domain S whose value is the area, change it into uv-coordinates, and evaluate to find the area.

XII. (6) Let D be the region in the in the xy-plane bounded by the triangle with vertices $(1,0)$, $(-1,0)$, and $(0,1)$. Partition D into four triangular regions using the lines $y = \frac{1}{2}$, $y = x$, and $y = -x$. Calculate the smallest and largest Riemann sums for the function $f(x,y) = y - x^2$ for this partition. Hint: It is rather easy to find maximum and minimum values of this function if you think about its level curves, especially the one that passes through $(\frac{1}{2}, \frac{1}{2})$.