Mathematics 2443-006H — Name (please print) ____________________

Final Examination

May 11, 2006

Instructions: Give brief, clear answers. Use theorems whenever possible.

I. (9) Let S be the portion of the cone $z = \sqrt{x^2+y^2}$ between $z=0$ and $z=2$. It can be parameterized by the formulas $x = u\cos(v)$, $y = u\sin(v)$, $z = u$.

(a) Sketch the domain R of this parameterization.

On the cone, we have $0 \le v \le 2\pi$, and since $z = u$, this portion of the cone corresponds to $0 \le u \le 2$. So R is the rectangle $0 \le v \le 2\pi$, $0 \le u \le 2$ in the uv-plane.

(b) Sketch the surface and some typical vectors $\vec{r}_u$ and $\vec{r}_v$.

[It is the standard cone between $z = 0$ and $z - 2$. Since v corresponds to the polar angle in the horizontal planes, r_v is tangent to the horizontal cross section circles and point to the right on the front side of the cone (when x and y are in their customary directions). Increasing u does not change the polar angle, but does increase z, so r_u points upward on the cone.

(c) Calculate $\vec{r}_u$ and $\vec{r}_v$ explicitly, and use them to calculate an upward normal vector to the surface.

Using the expression for x, y, and z in terms of u and v, we have $\vec{r}_u = \dfrac{\partial(u\cos(v))}{\partial u}\vec{\imath} + \dfrac{\partial(u\sin(v))}{\partial u}\vec{\jmath} + \dfrac{\partial u}{\partial u}\vec{k} = \cos(v)\vec{\imath} + \sin(v)\vec{\jmath} + \vec{k}$. Similarly, $\vec{r}_v = -u\sin(v)\vec{\imath} + u\cos(v)\vec{\jmath}$. Using the right-hand rule, the upward normal is $\vec{r}_v \times \vec{r}_u = -u\cos(v)\vec{\imath} - u\sin(v)\vec{\jmath} + u\vec{k}$.

(d) Express dS in terms of dR.

We have $\|\, \vec{r}_v \times \vec{r}_u \,\| = \|\, -u\cos(v)\vec{\imath} - u\sin(v)\vec{\jmath} + u\vec{k} \,\| = \sqrt{u^2\cos^2(v) + u^2\sin^2(v) + u^2} = \sqrt{2u^2} = \sqrt{2}\,u$, so $dS = \sqrt{2}\,u\,dR$.

II. (5) It is a fact that any simple closed loop C in 3-dimensional space bounds a two-sided surface S (although if C is knotted, S will not be a disk, but a more complicated surface), and Stokes' Theorem applies to any surface bounded by C, not just disks. Using this fact, together with Stokes' Theorem, verify that if C is any simple closed loop, $\displaystyle\int_C \nabla f \cdot d\vec{r} = 0$. (Of course, this follows from the Fundamental Theorem for Line Integrals as well.) Verify any facts about curl that may be needed in your argument.

Using Stokes' Theorem on a surface S bounded by C, we have $\displaystyle\int_C \nabla f \cdot d\vec{r} = \iint_S \operatorname{curl}(\nabla f)\cdot d\vec{S}$. But $\operatorname{curl}(\nabla f) = \operatorname{curl}(f_x\vec{\imath} + f_y\vec{\jmath} + f_z\vec{k}) = (f_{zy} - f_{yz})\vec{\imath} - (f_{zx} - f_{xz})\vec{\jmath} + (f_{yx} - f_{xy})\vec{k} = 0\vec{\imath} + 0\vec{\jmath} + 0\vec{k} = \vec{0}$, so $\displaystyle\iint_S \operatorname{curl}(\nabla f)\cdot d\vec{S} = \iint_S \vec{0}\cdot d\vec{S} = 0$.

III. (6) Verify that if S and E satisfy the hypotheses of the Divergence Theorem, then:

(a) the volume of E is $\frac{1}{3}\displaystyle\iint_S (x\vec{\imath} + y\vec{\jmath} + z\vec{k})\cdot d\vec{S}$.

By the Divergence Theorem, $\frac{1}{3}\displaystyle\iint_S (x\vec{\imath}+y\vec{\jmath}+z\vec{k})\cdot d\vec{S} = \frac{1}{3}\iiint_E \operatorname{div}(x\vec{\imath}+y\vec{\jmath}+z\vec{k})\,dV = \frac{1}{3}\iiint_E 3\,dV = \iiint_E dV$, which is the volume of E.

(b) $\iint_S D_{\vec{n}} f\ dS = \iiint_E \Delta f\, dV$, where $\vec{n}$ is the unit normal to the surfaces and Δf is the Laplacian $f_{xx}\vec{\imath} + f_{yy}\vec{\jmath} + f_{zz}\vec{k}$.

The Divergence Theorem gives $\iint_S D_{\vec{n}} f\ dS = \iint_S \nabla f \cdot \vec{n}\ dS = \iint_S \nabla f \cdot d\vec{S} = \iiint_E \operatorname{div}(\nabla f)\, dV$. Since $\operatorname{div}(\nabla f) = \operatorname{div}(f_x\vec{\imath} + f_y\vec{\jmath} + f_z\vec{k}) = f_{xx} + f_{yy} + f_{zz} = \Delta f$, the last integral equals $\iiint_E \Delta f\, dV$.

IV. (6) Use the Divergence Theorem to calculate $\iint_S (4x^3z\vec{\imath} + 4y^3z\vec{\jmath} + 3z^4\vec{k}) \cdot d\vec{S}$ where S is the boundary of the solid hemisphere $x^2 + y^2 + z^2 \le R^2$, $0 \le z$. Hint: use spherical coordinates on the solid hemisphere, and the fact that $x^2 + y^2 + z^2 = \rho^2$ to simplify the integrand.

Letting E be the solid hemisphere, the Divergence Theorem gives

$$\iint_S (4x^3z\vec{\imath} + 4y^3z\vec{\jmath} + 3z^4\vec{k}) \cdot d\vec{S} = \iiint_E 12x^2z + 12y^2z + 12z^3\ dV$$

$$= \iiint_E 12(x^2+y^2+z^2)z\ dV = \int_0^{2\pi}\int_0^{\pi/2}\int_0^R 12\rho^2\ \rho\cos(\phi)\ \rho^2\sin(\phi)\, d\rho\, d\phi\, d\theta$$

$$= \frac{12\,R^6}{6}\int_0^{2\pi}\int_0^{\pi/2}\cos(\phi)\sin(\phi)\, d\rho\, d\phi\, d\theta = 2R^6\int_0^{2\pi} \frac{\sin^2(\phi)}{2}\Big|_0^{\pi/2} d\theta = 2\pi\, R^6$$

V. (6) Use Stokes' Theorem to evaluate $\int_C (e^{-x}\vec{\imath} + e^x\vec{\jmath} + e^z\vec{k}) \cdot d\vec{r}$, where C is the boundary of the portion of the surface $x+y+z = 1$ that lies in the first octant. You may take as known the fact that $\operatorname{curl}(e^{-x}\vec{\imath} + e^x\vec{\jmath} + e^z\vec{k}) = e^x\vec{k}$, no need to calculate it.

Let S be the portion of the plane in the first octant. Using Stokes' Theorem, we have $\int_C (e^{-x}\vec{\imath} + e^x\vec{\jmath} + e^z\vec{k}) \cdot d\vec{r} = \iint_S \operatorname{curl}(e^{-x}\vec{\imath} + e^x\vec{\jmath} + e^z\vec{k}) \cdot d\vec{S} = \iint_S e^x\vec{k} \cdot d\vec{S}$. Now S is the graph of the function $z = 1-x-y$ over the domain R in the xy-plane given by $0 \le x \le 1$, $0 \le y \le 1-x$. Using the formula for a surface integral on a surface that is the graph of a function, we find $\iint_S e^x\vec{k} \cdot d\vec{S} = \iint_R e^x\, dR = \int_0^1\int_0^{1-y} e^x\, dx\, dy = \int_0^1 e^{1-y} - 1\, dy = (-e^{1-1} + e^{1-0}) - 1 = e - 2$.

VI. (5) Use implicit differentiation to calculate $\dfrac{\partial R}{\partial R_3}\Big|_{(R_1,R_2,R_3)=(\sqrt{3},\sqrt{6},2)}$ if $\dfrac{1}{R} = \dfrac{1}{R_1^2} + \dfrac{1}{R_2^2} + \dfrac{1}{R_3^2}$.

$-\dfrac{1}{R^2}\dfrac{\partial R}{\partial R_3} = 0 + 0 - \dfrac{2}{R_3^3}$, so $\dfrac{\partial R}{\partial R_3} = \dfrac{2R^2}{R_3^3}$. When $(R_1, R_2, R_3) = (\sqrt{3}, \sqrt{6}, 2)$, $R = 4/3$, so $\dfrac{\partial R}{\partial R_3}\Big|_{(R_1,R_2,R_3)=(\sqrt{3},\sqrt{6},2)} = \dfrac{2(\frac{4}{3})^2}{8} = \dfrac{4}{9}$.

VII. (6) If z is a function of x and y, calculate $\dfrac{\partial z}{\partial r}$ and $\dfrac{\partial z}{\partial \theta}$, where r and θ are the polar coordinates. Write each result in terms of $\dfrac{\partial z}{\partial x}$, $\dfrac{\partial z}{\partial y}$, x, y, and r, that is, without using θ explicitly.

We have $x = r\cos(\theta)$ and $y = r\sin(\theta)$, so using the Chain Rule gives

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial(r\cos(\theta))}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial(r\sin(\theta))}{\partial r} = \cos(\theta)\frac{\partial z}{\partial x} + \sin(\theta)\frac{\partial z}{\partial y} = \frac{1}{r}\Big(x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}\Big)$$

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\frac{\partial(r\cos(\theta))}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial(r\sin(\theta))}{\partial \theta} = -r\sin(\theta)\frac{\partial z}{\partial x} + r\cos(\theta)\frac{\partial z}{\partial y} = -y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y}$$

VIII. Calculate each of the following.
(12) (a) The directional derivative of $\dfrac{1}{xy} + \dfrac{1}{yz}$ at $(2, 1, 2)$ in the direction toward the origin.

We have $\nabla\Big(\dfrac{1}{xy} + \dfrac{1}{yz}\Big) = -\dfrac{1}{x^2y}\vec{\imath} - \Big(\dfrac{1}{xy^2} + \dfrac{1}{y^2z}\Big)\vec{\jmath} - \dfrac{1}{yz^2}\vec{k}$. so the gradient at $(2,1,2)$ is $-\frac{1}{4}\vec{\imath} - \vec{\jmath} - \frac{1}{4}\vec{k}$. A vector in the direction of the origin is $-2\vec{\imath} - j - 2\vec{k}$, whose length is 3, so a unit vector in the direction of the origin is $-\frac{2}{3}\vec{\imath} - \frac{1}{3}j - \frac{2}{3}\vec{k}$. Taking the dot product of the gradient vector with this unit vector gives $\frac{2}{3}$.

(b) The maximum rate of change of $qe^{-p} - pe^{-q}$ at $(p, q) = (0, 0)$, and the direction in which it occurs.

We calculate $\nabla(qe^{-p} - pe^{-q}) = (-qe^{-p} - e^{-q})\vec{\imath} + (e^{-p} + pe^{-q})\vec{\jmath}$. At this origin, this is $-\vec{\imath} + \vec{\jmath}$, so this is the direction of the maximum rate of change, and this maximum is $\| - \vec{\imath} + \vec{\jmath}\| = \sqrt{2}$.

(c) A vector-valued function giving the line perpendicular to the level surface of xyz at the point $(1, 2, 3)$.

We calculate $\nabla(xyz) = yz\vec{\imath} + xz\vec{\jmath} + xy\vec{k}$, whose value at $(1, 2, 3)$ is $6\vec{\imath} + 3\vec{\jmath} + 2\vec{k}$. This is a direction vector for the normal line, which is then given by the vector-valued function $\vec{r}(t) = (1 + 6t)\vec{\imath} + (2 + 3t)\vec{\jmath} + (3 + 2t)\vec{k}$.

(d) An equation for the tangent plane to the level surface of $\dfrac{1}{xyz}$ at the point $(1, 2, 3)$.

The level surfaces of $\dfrac{1}{xyz}$ are the same as those of the function xyz. We already calculated the gradient vector of xyz at this point to be $6\vec{\imath} + 3\vec{\jmath} + 2\vec{k}$, and it is a normal vector to the tangent plane. So an equation for the tangent plane is $6(x - 1) + 3(y - 2) + 2(z - 3) = 0$, or $6x + 3y + 2z = 18$.

IX. Six positive numbers x, y, z, u, v, and w, each less than or equal to 2, are multiplied together. Use
(5) differentials to estimate the maximum possible error in the computed product that might result from rounding each number off to the nearest whole number.

We calculate $d(xyzuvw) = yzuvw\,dx + xzuvw\,dy + xyuvw\,dz + xyzvw\,du + xyzuw\,dv + xyzuv\,dw$. Rounding off to the nearest integer allows any of dx, etc., to be as large as 0.5, and each of the five-term products is at most $2^5 = 32$, so the linear part of the error is no more than $6 \cdot 32 \cdot 0.5 = 96$.

X. Six positive numbers x, y, z, u, v, and w, are multiplied together. The first three are increasing at 0.5
(5) units per second, while the last three are decreasing at 0.1 units per second. Find the rate of change of the product at a moment when all of the numbers except w equal 1, and $w = 2$.

The Chain Rule gives

$$\frac{d(xyzuvw)}{dt} = yzuvw\frac{dx}{dt} + xzuvw\frac{dy}{dt} + xyuvw\frac{dz}{dt} + xyzvw\frac{du}{dt} + xyzuw\frac{dv}{dt} + xyzuv\frac{dw}{dt}\ .$$

Specializing to a moment when each of x, etc. equals 1, we obtain

$$2 \cdot 0.5 + 2 \cdot 0.5 + 2 \cdot 0.5 - 2 \cdot 0.1 - 2 \cdot 0.1 - 1 \cdot 0.1 = 2.5\ .$$

XI. Calculate the area inside the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 3$ as follows.
(7) (a) Let S be the region bounded by the ellipse. Define ϕ from the uv-plane to the xy-plane by $\phi(u,v) = (au, bv)$. Determine the region R in the uv-plane that corresponds to S under ϕ.

In (u,v)-coordinates, the boundary ellipse becomes $\dfrac{(au)^2}{a^2} + \dfrac{(bv)^2}{b^2} = 3$, that is, $u^2 + v^2 = 1$, so R is the disk of radius $\sqrt{3}$.

(b) Calculate the Jacobian $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{pmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} \end{pmatrix}$ and its determinant.

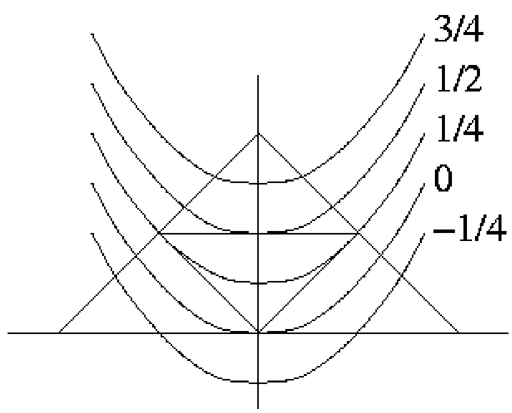
$\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, whose determinant is ab.

(c) Write a double integral over the domain S whose value is the area, change it into uv-coordinates, and evaluate to find the area.

$$\iint_S dS = \iint_R ab\, dR = ab \iint_R dR = ab \cdot (\text{area of } R) = 3\pi ab.$$

XII. Let D be the region in the in the xy-plane bounded by the triangle with vertices $(1,0)$, $(-1,0)$, and $(0,1)$.
(6) Partition D into four triangular regions using the lines $y = \frac{1}{2}$, $y = x$, and $y = -x$. Calculate the smallest and largest Riemann sums for the function $f(x,y) = y - x^2$ for this partition. Hint: It is rather easy to find maximum and minimum values of this function if you think about its level curves, especially the one that passes through $(\frac{1}{2}, \frac{1}{2})$.

This figure shows the partitioned region and some level curves of the function $y - x^2$.



Notice that the level curve where the function equals $1/4$ is exactly tangent to $y = x$ at $(\frac{1}{2}, \frac{1}{2})$ (since the slope of $y - x^2$ is $2x$), so the maximum value on the right and left triangles is $1/4$, at the points $(\pm\frac{1}{2}, \frac{1}{2})$. The maximum on the center triangle is $1/2$ at the point $(0, \frac{1}{2})$, and the maximum value on the top triangle is 1 at $(0,1)$. Since the area of each triangle is $1/4$, the maximum Riemann sum is $\frac{1}{4}\big(1 + 1/2 + 1/4 + 1/4\big) = 1/2$. Similarly, the minimum value on top triangle is $1/4$ at $(\pm 1/2, 1/2)$, the minimum on the middle triangle is 0 at $(0,0)$, and on the the right and left triangles is -1, at the points $(\pm 1, 0)$, so the minimum Riemann sum is $\frac{1}{4}\big(1/4 + 0 + (-1) + (-1)\big) = -\frac{7}{16}$.