Mathematics 2443-006H Name (please print) ____________________

Examination III

April 27, 2006

Instructions: Give brief, clear answers. Use Green's Theorem whenever possible.

I. (6) Let $\vec{F}$ be the vector field $\dfrac{-y}{x^2+y^2}\vec{\imath}+\dfrac{x}{x^2+y^2}\vec{\jmath}$. Verify by calculation (direct or indirect) that $\int_C \vec{F}\cdot d\vec{r}$ is not path-independent on the domain $\{(x,y) \mid (x,y)\neq(0,0)\}$.

Take C to be the unit circle, on which $\vec{F}$ is simply $-y\vec{\imath}+x\vec{\jmath}$. Here it is the unit tangent vector, since it has length $\sqrt{(-y)^2+x^2}=1$ and is perpendicular to the position vectors $x\vec{\imath}+y\vec{\jmath}$ which are radii of the circle. So $\int_C \vec{F}\cdot d\vec{r}=\int_C \vec{F}\cdot\vec{T}\,ds=\int_C ds=2\pi$. Since C is a closed loop, but $\int_C \vec{F}\cdot d\vec{r}$ is not 0, $\int_C \vec{F}\cdot d\vec{r}$ is not path-independent. [Of course, one can also calculate directly by parameterizing C as $\vec{r}(t)=\cos(t)\vec{\imath}+\sin(t)\vec{\jmath}$.]

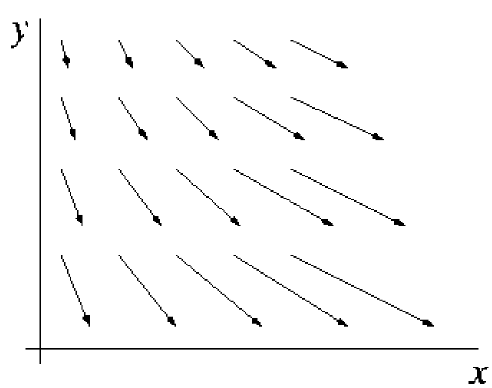
II. (12) (a) Evaluate the line integral $\int_C xy\,dx+x^2y\,dy$ directly, where C is the triangle with vertices $(0,0)$, $(1,0)$, and $(1,2)$.

Let C_1 be the line segment from $(0,0)$ to $(1,0)$, C_2 the line segment from $(1,0)$ to $(1,2)$, and C_3 the line segment from $(1,2)$ to $(0,0)$. On C_1, $y=0$ so the integrand is 0 and therefore the integral is 0. On C_2, putting $x=1$ and $y=t$ for $0\leq t\leq 2$, we have $dx=0\,dt$, $dy=dt$, so $\int_{C_2} xy\,dx+x^2y\,dy=\int_0^2 0+t\,dt=2$. Parameterizing $-C_3$ by $x=t$ and $y=2t$, we have $\int_{C_3} xy\,dx+x^2y\,dy=-\int_0^1 t\cdot 2t+t^2\cdot 2t\cdot 2\,dt=-\dfrac{2}{3}-1=-\dfrac{5}{3}$. Therefore $\int_C xy\,dx+x^2y\,dy=\dfrac{1}{3}$.

(b) Evaluate it using Green's Theorem.

For the triangle T bounded by C, $\int_C xy\,dx+x^2y\,dy=\iint_T \dfrac{\partial}{\partial x}(x^2y)-\dfrac{\partial}{\partial y}(xy)\,dA=\iint_T 2xy-x\,dA=\int_0^1\int_0^{2x} 2xy-x\,dy\,dx=\int_0^1 xy^2-xy\Big|_0^{2x}\,dx=\int_0^1 4x^3-2x^2\,dx=1-\dfrac{2}{3}=\dfrac{1}{3}$.

III. (7) The figure to the right shows a vector field $P\vec{\imath}+Q\vec{\jmath}$ on a portion of the plane. Based on its appearance there:



(a) Explain geometrically why $\dfrac{\partial P}{\partial x}$ is positive.

As you move to the right, the horizontal components are increasing.

(b) Explain geometrically why $\dfrac{\partial P}{\partial y}$ is negative.

As you move upward, the horizontal components are decreasing.

(c) Explain geometrically why $\dfrac{\partial Q}{\partial x}$ is zero.

As you move to the right, the vertical components do not change.

(d) Explain geometrically why $\dfrac{\partial Q}{\partial y}$ is positive.

As you move upward, the vertical components, although negative, are nonetheless increasing.

(e) Determine whether $\operatorname{div}(P\vec{\imath}+Q\vec{\jmath})$ is positive or negative.

$\operatorname{div}(P\vec{\imath}+Q\vec{\jmath}) = \dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y}$ is the sum of two positive terms.

(f) Determine whether $\operatorname{curl}(P\vec{\imath}+Q\vec{\jmath})\cdot\vec{k}$ is positive or negative.

$\operatorname{curl}(P\vec{\imath}+Q\vec{\jmath})\cdot\vec{k} = \dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}$, with the first term 0 and $\dfrac{\partial P}{\partial x}$ negative, so $\operatorname{curl}(P\vec{\imath}+Q\vec{\jmath})\cdot\vec{k}$ is positive.

IV. (6) Let f be a scalar function of three variables and let $\vec{F}$ be a vector field on a 3-dimensional domain. Writing $\vec{F}$ as $P\vec{\imath}+Q\vec{\jmath}+R\vec{k}$, verify that $\operatorname{div}(f\vec{F}) = f\operatorname{div}(\vec{F}) + \vec{F}\cdot\nabla f$.

$\operatorname{div}(f\vec{F}) = \operatorname{div}(fP\vec{\imath}+fQ\vec{\jmath}+fR\vec{k}) = (fP)_x+(fQ)_y+(fR)_z = f_xP+fP_x+f_yQ+fQ_y+f_zR+fR_z = f(P_x+Q_y+R_z)+(P\vec{\imath}+Q\vec{\jmath}+R\vec{k})\cdot(f_x\vec{\imath}+f_y\vec{\jmath}+f_z\vec{k}) = f\operatorname{div}(\vec{F})+\vec{F}\cdot\nabla f$.

V. (6) (a) Sketch the vector field $\dfrac{-y}{x^2+y^2}\,\vec{\imath}+\dfrac{x}{x^2+y^2}\,\vec{\jmath}$.

(see your class notes)

(b) Explain the important phenomenon (related to Clairaut's Theorem) that the vector field $\dfrac{-y}{x^2+y^2}\,\vec{\imath}+\dfrac{x}{x^2+y^2}\,\vec{\jmath}$ illustrates.

It satisfies the necessary condition $\dfrac{\partial Q}{\partial y} = \dfrac{\partial P}{\partial x}$ to be conservative, but (as seen in problem I above) it is not conservative on its domain $\mathbb{R}^2 - \{(0,0)\}$. It illustrates that the hypothesis that the domain is simply-connected is needed for the necessary condition to be sufficient.

VI. (5) Let D be a connected planar domain.

(a) Define what it means to say that D is simply-connected (do better than "no holes").

If C is any simple (no self-crossings) loop in D, then every point in the plane that is enclosed by C is also in D.

(b) State the theorem discussed in class that uses the hypothesis that D is simply-connected.

If D is simply-connected and $P\vec{\imath}+Q\vec{\jmath}$ is a vector field on D that satisfies $\dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y}$, then $P\vec{\imath}+Q\vec{\jmath}$ is conservative.

VII. Suppose that $\vec{F} = P\vec{\imath} + Q\vec{\jmath}$ is a vector field on the plane and let C be the unit circle. Suppose that at
(6) points on C (but not necessarily on the rest of D), $\vec{F}(x,y) = x\vec{\imath} + y\vec{\jmath}$.

(a) Verifty that on C, $\vec{F}$ equals the outward unit normal $\vec{n}$.

Since $\vec{F}$ is the position vector of (x,y), that is, it looks like a radius of the circle, it is perpendicular to the circle at the point (x,y). Therefore it is normal and points outward. Also, on C it has length $\sqrt{x^2+y^2} = 1$, so it has unit length.

(b) Calculate $\displaystyle\iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA$.

By the Tangential Form for Green's Theorem, we have $\displaystyle\iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA = \iint_D \operatorname{curl}(\vec{F}) \cdot \vec{k}\, dA = \int_C \vec{F} \cdot \vec{T}\, ds = \int_C 0\, ds = 0$.

(c) Calculate $\displaystyle\iint_D \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\, dA$.

By the Normal Form for Green's Theorem, we have $\displaystyle\iint_D \frac{\partial P}{\partial x} + \frac{\partial P}{\partial y}\, dA = \int_C \vec{F} \cdot \vec{n}\, ds = \int_C ds = 2\pi$.

VIII. Calculate $\displaystyle\int_C (y\vec{\imath} - x\vec{\jmath}) \cdot d\vec{r}$, where C is the equilateral triangle that has one side equal to the straight line
(6) from $(1,1)$ to $(201,1)$ but does not lie completely in the first quadrant.

Let T be the triangle enclosed by C. Using Green's Theorem, $\displaystyle\int_C y\vec{\imath} \cdot d\vec{r} = \iint_T \frac{\partial(-x)}{\partial x} - \frac{\partial(y)}{\partial y}\, dA = \iint_T -2\, dA$, which is -2 times the area of T. Since T has base of length 200, a little geometry shows that the area of T is $(\sqrt{3}/2)(200)^2/2 = 10{,}000\sqrt{3}$, so the answer is $-20{,}000\sqrt{3}$.