Mathematics 2513-001 Name (please print) ______________________

Final Examination Form A

May 9, 2006

Instructions: Give brief, clear answers. "Prove" means "give an argument".

I. (5) Let $T(s, c)$ be "Student s has taken course c." Write each of the following statements in logical notation, putting in all necessary quantifiers using the sets $\mathcal{S}$ of all students and $\mathcal{C}$ of all courses. If your answer involves a negation, simplify as much as possible.

(a) Ann has taken both Diffy Q and Linear.

$T(\text{Ann}, \text{Diffy Q}) \wedge T(\text{Ann}, \text{Linear})$

(b) Everyone has taken English Comp.

$\forall s \in \mathcal{S}, T(s, \text{English Comp})$

(c) No student has taken every course.

$\neg \exists s \in \mathcal{S}, \forall c \in \mathcal{C}, T(s, c)$. This simplifies to $\forall s \in \mathcal{S}, \exists c \in \mathcal{C}, \neg T(s, c)$.

(d) Someone (some one person) has taken all the courses that I have taken.

$\exists s \in S, \forall c \in \mathcal{C}, T(\text{I}, c) \Rightarrow T(s, c)$

(e) Phillip has not taken any of the courses that Ann has.

$\forall c \in \mathcal{C}, T(\text{Ann}, c) \Rightarrow \neg T(\text{Phillip}, c)$

II. (8) Assuming that $y \colon A \to B$, give precise definitions of the following, using logical notation and/or set notation as appropriate. Remember that the domain and codomain are part of the definition of a function, so must be specified when you are defining a function.

(a) the range of y

The range of y is $\{b \in B \mid \exists a \in A, y(a) = b\}$, or $\{y(a) \mid a \in A\}$.

(b) the preimage of an element b of B

The preimage of b is $\{a \in A \mid y(a) = b\}$.

(c) the inverse function y^{-1}, assuming that y is bijective

$y^{-1} \colon B \to A$ is defined by $y^{-1}(b) = a \Leftrightarrow y(a) = b$.

(d) $y = x$

$x \colon A \to B$ and $\forall a \in A, x(a) = y(a)$.

(e) the graph of y

The graph of y is the set $\{(a, y(a)) \mid a \in A\}$ (or $\{(a, b) \in A \times B \mid y(a) = b\}$).

III. (4) Define $\gcd(a, b)$. Define *relatively prime*. Explain why the integer 1 is relatively prime to every other integer.

For integers a and b, not both 0, $\gcd(a, b)$ is the largest integer that divides both a and b. We say that a and b are relatively prime when $\gcd(a, b) = 1$. For any integer b, $\gcd(1, b) = 1$ because the largest integer that divides 1 is 1, and 1 always divides b, so 1 must be the largest common divisor.

IV. (4) Use proof by contradiction to prove that the sum of a rational number and an irrational number must be irrational.

Suppose for contradiction that there exist a rational number x and an irrational number y for which $x+y$ is rational. Write $x = \dfrac{p}{q}$ and $x+y = \dfrac{r}{s}$ for some integers p, q, r, and s. We calculate that $y = (x+y) - x = \dfrac{rq-ps}{qs}$, so y is also rational, which is a contradiction.

V. (10) Tell how many strings of six letters satisfy each of the following conditions. Make reasonable simplifications, but leave products in factored form (that is, do not multiply them out).

(a) contain no repeated letter.

There are 26 choices for the first letter, and for each such choice there are 25 choices for the second, and so on, so by the Product Rule, there are $26 \cdot 25 \cdot \ldots \cdot 21$ possible choices of such strings. [Or, one can simply say this is the number of 6-permutations of 26 elements, so is $26 \cdot 25 \cdot \ldots \cdot 21$.]

(b) start with z or end with z, but do not both start and end with z.

First count the number that start with z and do not end with z. There are 26 choices for the second through fifth letters, and 25 choices for the last letter, so the Product Rule gives $25 \cdot (26)^4$ possibilities. Similarly, there are $25 \cdot (26)^4$ strings that end with an z but do not start with one. These do not occur at the same time, so by the Sum Rule, there are $2 \cdot 25 \cdot (26)^4$ strings meeting one of the two conditions.

(c) contain exactly one vowel

First count the number with a vowel in the first place. There are 5 choices for the first letter, and 21 for the second through sixth letters, so the Product Rule gives $5 \cdot (21)^5$ possibilities. Similarly, there are $5 \cdot (21)^5$ strings with one vowel in the remaining five places. None of these occur at the same time, so by the Sum Rule, there are $6 \cdot 5 \cdot (21)^5$ strings with exactly one vowel.

(d) contain no immediate repeat (for example, no “bb”), although a letter can repeat later (“bab” can appear).

There are 26 choices for the first letter. For each of these choices, there are 25 choices of second letter (excluding only the one in the first place), then there are 25 choices of third letter (excluding only the one in the second place), and so on, giving $26 \cdot (25)^5$ possibilities.

(e) have consonants in the first, third, and fifth position and vowels in the second, fourth, and sixth positions, and have no repeated consonant, although they may contain repeated vowels.

There are 21 choices for the first letter. For each of these, there are 5 choices of second letter. For each of these, there are only 20 choices in the third (since consonants may not be repeated), but again there are 5 choices for the fourth (since vowels consonants may be repeated. Continuing, we have 19 choices for the fifth place, and 5 for the sixth, giving $5^3 \cdot 21 \cdot 20 \cdot 19$ possibilities.

VI. (3) State the (basic, not generalized) Pigeonhole Principle.

If $k+1$ objects are place in k boxes, then at least one box contains at least two objects.

VII. (4) Prove that the function $f\colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}\times\mathbb{R}$ defined by $f(x,y) = (2x+y, x+y)$ is injective.

Let (x_1, y_1) and (x_2, y_2) be elements of $\mathbb{R}\times\mathbb{R}$, and assume that $f(x_1, y_1) = f(x_2, y_2)$. This says that $(2x_1+y_1, x_1+y_1) = (2x_2+y_2, x_2+y_2)$, which means that $2x_1+y_1 = 2x_2+y_2$ and $x_1+y_1 = x_2+y_2$. Subtracting these two equations gives $x_1 = x_2$. The second equation becomes $x_1+y_1 = x_1+y_2$, giving $y_1 = y_2$. Since $x_1 = x_2$ and $y_1 = y_2$, we have $(x_1, y_1) = (x_2, y_2)$.

VIII. (8) Determine the number of elements in each of the following sets. If binomial coefficients appear, write them as quotients of factorials, but do not multiply out the factorials.

(a) $\mathcal{P}(\{1,2\}\times\{1,3\})$

Since $\{1,2\}$ and $\{1,3\})$ each have 2 elements, $\{1,2\}\times\{1,3\}$ has 4 elements. Therefore its power set $\mathcal{P}(\{1,2\}\times\{1,3\})$ has 2^4 elements.

(b) $\mathcal{P}(\{2\}\cup\mathcal{P}(\{1,2\}\times\{1,3\}))$

We already saw that $\mathcal{P}(\{1,2\}\times\{1,3\})\})$ has $2^4 = 16$ elements so $\{2\}\cup\mathcal{P}(\{1,2\}\times\{1,3\})$ has 17 elements. Therefore its power set $\mathcal{P}(\{2\}\cup\mathcal{P}(\{1,2\}\times\{1,3\}))$ has 2^{17} elements.

(c) $\{S \subseteq \{1,2,3,\ldots,50\} \mid S \text{ has cardinality } 2\}$

This is the number of 2-element subsets of a set of 50 elements, so there are $\binom{50}{2} = \frac{50!}{2!\,48!}$ of them. [This equals $\frac{50\cdot 49}{2} = 49\cdot 25$.]

(d) $\{s \mid s$ is a bit string of length 20 containing either exactly five 0's or exactly fifteen 0's$\}$ (recall that a bit string is a finite sequence of 0's and 1's).

The number of bit strings with five 0's is the number of five-element subsets of 20 elements (because choosing the places for the 0's amounts to choosing a five-element subset of the 20 places), so there are $\binom{20}{5}$ with five 0's. Similarly, there are $\binom{20}{15}$ with fifteen 0's. Since no bit strings have both exactly five 0 and exactly fifteen 0's, the Sum Rule says the total is $\binom{20}{5}+\binom{20}{15} = \binom{20}{5}+\binom{20}{5} = 2\,\frac{20!}{5!\,15!}$. [This equals $\frac{2\cdot 20\cdot 19\cdot 18\cdot 17\cdot 16}{5\cdot 4\cdot 3\cdot 2} = \frac{19\cdot 18\cdot 17\cdot 16}{3} = 19\cdot 17\cdot 16\cdot 6$.]

IX. (2) Write the following as an implication: "$x^2 \geq 2$ for at most one x".

$(x^2 \geq 2 \wedge y^2 \geq 2) \Rightarrow x = y$

X. (4) Let $f\colon X \to Y$ and $g\colon Y \to Z$. Prove that if f and g are surjective, then the composition $g\circ f$ is surjective.

Assume that f and g are surjective. Let $z \in Z$. Since g is surjective, there exists $y \in Y$ such that $g(y) = z$. Since f is surjective, there exists $x \in X$ such that $f(x) = y$. Then, we have $(g\circ f)(x) = g(f(x)) = g(y) = z$.

XI. (4) Prove that $2^n < n!$ whenever $n \geq 4$.

We will argue by induction starting at 4. For $n = 4$, $2^4 = 16$ and $4! = 24$, so $2^4 < 4!$. For the inductive step, let $k \geq 4$ and assume that $2^k < k!$. Then we have $2^{k+1} = 2\cdot 2^k < 2\cdot k! < (k+1)\cdot k! = (k+1)!$. By induction, $2^n < n!$ for all $n \geq 4$.

XII. (4) Let X be the set of all infinite sequences in which each term is an integer, that is, all sequence $z_1z_2z_3\cdots$, where each $z_i \in \mathbb{Z}$. Using Cantor's idea, prove that X is not countable.

Suppose for contradiction that X is countable, so that there exists a bijective function $f\colon \mathbb{N} \to X$. Write

$$\begin{aligned} f(1) &= z_{11}z_{12}z_{13}z_{14}\cdots \\ f(2) &= z_{21}z_{22}z_{23}z_{24}\cdots \\ f(3) &= z_{31}z_{32}z_{33}z_{34}\cdots \\ f(4) &= z_{41}z_{42}z_{43}z_{44}\cdots \\ &\vdots \end{aligned}$$

For each i, define $z_i = 0$ if $z_{ii} \neq 0$ and $z_i = 1$ if $z_{ii} = 0$. Let z be the sequence $z_1z_2z_3z_4\cdots$, an element of X. For each n, z differs from $f(n)$ in at least the n'th place, since $z_n \neq z_{nn}$, so $z \neq f(n)$. Thus z is an element of X that is not in the range of f, so f is not surjective. This contradicts the fact that f was bijective.

XIII. (6) Let a, b, c, m, and n be integers.
(a) Using the definition of "divides", prove that if $a|b$ and $a|c$, then $a|b+c$.

Assume that $a|b$ and $a|c$. Then, there exist integers k and ℓ so that $b = ka$ and $c = \ell a$, so $b + c = ka + \ell a = (k + \ell)a$. Therefore $a|b + c$.

(b) Using the definition of "divides", prove that if $a|b$, then $a|mb$.

Assume that $a|b$. Then, there exists an integer k so that $b = ka$. Then, $mb = (mk)a$, so $a|mb$.

(c) Give a step-by-step argument using (a) and (b) to deduce: If $a|b$ and $a|c$, then $a|mb + nc$.

Assume that $a|b$ and $a|c$. By (b), $a|mb$ and $a|nc$. By (a), $a|mb + nc$.

XIV. (4) Let A be a countable set, so that A can be written as $\{a_1, a_2, a_3, \ldots\}$, and let $B = \{b_1, b_2, b_3, \ldots\}$ be another countable set.
(a) Write down a list (at least, the first few elements of such a list) of all elements of $A \times B$ whose first coordinate is a_1.

$(a_1, b_1), (a_1, b_2), (a_1, b_3), \ldots$

(b) Prove that $A \times B$ is countable.

Form a list of lists in which the n^{th} list gives all the elements of $A \times B$ whose first element is a_n:

$$\begin{gathered} (a_1, b_1), (a_1, b_2), (a_1, b_3), \ldots \\ (a_2, b_1), (a_2, b_2), (a_2, b_3), \ldots \\ (a_3, b_1), (a_3, b_2), (a_3, b_3), \ldots \\ \vdots \end{gathered}$$

Using Cantor's diagonals trick, we can make this into a single list of all elements of $A \times B$:

$$(a_1, b_1),\ \ (a_2, b_1), (a_1, b_2),\ \ (a_3, b_1), (a_2, b_2), (a_1, b_3),\ \ (a_4, b_1)\ldots$$

and then a bijection $f\colon \mathbb{N} \to A \times B$ is given by sending n to the n^{th} element on this list.

XV. State the Fundamental Theorem of Arithmetic.
(2)

Any integer greater than 1 can be written as a product of prime factors. If the factors are written in nondecreasing order, then this factorization is unique.

XVI. How many subsets of a set with 300 elements contain more than one element?
(3)

There are 2^{300} subsets in total. Of these, one contains no elements, and 300 contain exactly one element, so the number with at least two elements is $2^{300} - 301$.