Mathematics 2513-001 Name (please print) ______________________

Final Examination Form B

May 9, 2006

Instructions: Give brief, clear answers. "Prove" means "give an argument".

I. (4) Use proof by contradiction to prove that the sum of a rational number and an irrational number must be irrational.

II. (10) Tell how many strings of six letters satisfy each of the following conditions. Make reasonable simplifications, but leave products in factored form (that is, do not multiply them out).
(a) contain no repeated letter.
(b) start with x or end with x, but do not both start and end with x.
(c) contain exactly one vowel
(d) contain no immediate repeat (for example, no "bb"), although a letter can repeat later ("bab" can appear).
(e) have vowels in the first, third, and fifth position and consonants in the second, fourth, and sixth positions, and have no repeated consonant, although they may contain repeated vowels.

III. (5) Let $T(s, c)$ be "Student s has taken course c." Write each of the following statements in logical notation, putting in all necessary quantifiers using the sets $\mathcal{S}$ of all students and $\mathcal{C}$ of all courses. If your answer involves a negation, simplify as much as possible.
(a) Phillip has taken both Diffy Q and Linear.
(b) Everyone has taken English Comp.
(c) No student has taken every course.
(d) Someone (some one person) has taken all the courses that I have taken.
(e) Laura has not taken any of the courses that Ann has.

IV. (8) Assuming that $x\colon A \to B$, give precise definitions of the following, using logical notation and/or set notation as appropriate. Remember that the domain and codomain are part of the definition of a function, so must be specified when you are defining a function.
(a) the range of x
(b) the preimage of an element b of B
(c) the inverse function x^{-1}, assuming that x is bijective
(d) $x = z$
(e) the graph of x

V. (4) Define $\gcd(a, b)$. Define *relatively prime*. Explain why the integer 1 is relatively prime to every other integer.

VI. (4) Prove that the function $f\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ defined by $f(x, y) = (2x + y, x + y)$ is injective.

VII. (8) Determine the number of elements in each of the following sets. If binomial coefficients appear, write them as quotients of factorials, but do not multiply out the factorials.
(a) $\mathcal{P}(\{1, 2\} \times \{1, 3\})$
(b) $\mathcal{P}(\{2\} \cup \mathcal{P}(\{1, 2\} \times \{1, 3\}))$
(c) $\{S \subseteq \{1, 2, 3, \ldots, 100\} \mid S \text{ has cardinality } 2\}$
(d) $\{s \mid s$ is a bit string of length 20 containing either exactly five 0's or exactly fifteen 0's$\}$ (recall that a bit string is a finite sequence of 0's and 1's).

VIII. (3) State the (basic, not generalized) Pigeonhole Principle.

IX. (2) Write the following as an implication: "$x^2 \geq 2$ for at most one x".

X. (4) Let $f\colon X \to Y$ and $g\colon Y \to Z$. Prove that if f and g are surjective, then the composition $g \circ f$ is surjective.

XI. (4) Prove that $2^n < n!$ whenever $n \geq 4$.

XII. (4) Let X be the set of all infinite sequences in which each term is an integer, that is, all sequence $z_1 z_2 z_3 \cdots$, where each $z_i \in \mathbb{Z}$. Using Cantor's idea, prove that X is not countable.

XIII. (4) Let A be a countable set, so that A can be written as $\{a_1, a_2, a_3, \ldots\}$, and let $B = \{b_1, b_2, b_3, \ldots\}$ be another countable set.

(a) Write down a list (at least, the first few elements of such a list) of all elements of $A \times B$ whose first coordinate is a_1.

(b) Prove that $A \times B$ is countable.

XIV. (6) Let a, b, c, m, and n be integers.

(a) Using the definition of "divides", prove that if $a|b$ and $a|c$, then $a|b + c$.

(b) Using the definition of "divides", prove that if $a|b$, then $a|mb$.

(c) Give a step-by-step argument using (a) and (b) to deduce: If $a|b$ and $a|c$, then $a|mb + nc$.

XV. (2) State the Fundamental Theorem of Arithmetic.

XVI. (3) How many subsets of a set with 200 elements contain more than one element?