Mathematics 2513-001 Name (please print) ____________________

Examination I Form B

February 14, 2006

Instructions: Give brief, clear answers.

I. (3) Using step-by-step logic, simplify the following expression: $\neg(\neg Q \wedge \forall z, (\ R(z) \Rightarrow P(z)\ )$

$$\neg(\neg Q \wedge \forall z, (\ R(z) \Rightarrow P(z)\ ) \equiv Q \vee \neg\forall z, (\ R(z) \Rightarrow P(z)\ )$$
$$\equiv Q \vee \exists z, \neg(\ R(z) \Rightarrow P(z)\ ) \equiv Q \vee \exists z, \neg(\ \neg R(z) \vee P(z)\ )$$
$$\equiv Q \vee \exists z, (\ R(z) \wedge \neg P(z)\ )$$

II. (2) Write the following as an implication: "$b^2 \geq 2$ for at most one b".

$(b^2 \geq 2 \wedge c^2 \geq 2) \Rightarrow b = c$

III. (4) Use a truth table to verify the tautology $\big(\neg Y \Rightarrow (X \wedge \neg X)\big) \Rightarrow Y$.

Y	X	$\neg Y$	$\neg X$	$X \wedge \neg X$	$\neg Y \Rightarrow (X \wedge \neg X)$	$\big(\neg Y \Rightarrow (X \wedge \neg X)\big) \Rightarrow Y$
T	T	F	F	F	T	T
T	F	F	T	F	T	T
F	T	T	F	F	F	T
F	F	T	T	F	F	T

IV. (3) Write each of the following as either $A \Rightarrow B$ or $B \Rightarrow A$:

(i) whenever A, B

$A \Rightarrow B$

(ii) A is necessary for B

$B \Rightarrow A$

(iii) A, when B

$B \Rightarrow A$

V. (3) Write the following statement in logical notation (and simplified so that it does not involve the negation symbol $\neg$) using the universal set $\mathcal{U} = \mathbb{Z}$: There is a positive integer that is not the sum of the squares of three integers.

$$\exists n, (n > 0 \wedge \neg(\exists a, \exists b, \exists c, a^2 + b^2 + c^2 = n))$$
$$\equiv \exists n, (n > 0 \wedge \forall a, \forall b, \forall c, \neg(a^2 + b^2 + c^2 = n))$$
$$\equiv \exists n, (n > 0 \wedge \forall a, \forall b, \forall c, a^2 + b^2 + c^2 \neq n)$$

VI. Give the general form of a proof by contradiction. That is, if the statement to be proven is P, give the main
(4) steps in the logical structure of the proof. Briefly explain why the argument proves the original assertion.

The general form is:

Statement: P.
proof: Assume $\neg P$.
...
Therefore Q.
But Q is false.
Therefore P. □

The first section of the argument proves the implication $\neg P \Rightarrow Q$, by a direct argument. This is equivalent to its contrapositive, $\neg Q \Rightarrow P$. Then, one observes that $\neg Q$ is true, so the implication $\neg Q \Rightarrow P$ guarantees that P is true.

VII. In this problem, you may take as known the fact that $\sqrt{2}$ is irrational.
(9) (a) Prove that the difference of two rational numbers must be rational (that is, that if x and y are rational, then $x - y$ is rational).

Let x and y be rational numbers. Then, we can write $x = \frac{p}{q}$ and $y = \frac{r}{s}$ for some integers p, q, r, and s. We calculate that $x - y = \frac{ps - rq}{qs}$, so $x - y$ is also rational.

(b) Prove that the sum of a rational number and an irrational number must be irrational.

Suppose for contradiction that there exist a rational number x and an irrational number y for which $x + y$ is rational. Using part (a), $y = (x + y) - x$ is rational, which is a contradiction.

(c) Give a counterexample to: The difference of two irrational numbers must be irrational.

Put $x = \sqrt{2}$ and $y = \sqrt{2}$. Then $x - y = 0$, which is rational.

VIII. Let $M(p, m)$ be "Person p has seen the movie m." Write each of the following statements in logical
(5) notation, putting in all necessary quantifiers using the sets $\mathcal{P}$ of all people and $\mathcal{M}$ of all movies. If your answer involves a negation, simplify as much as possible.
(a) Jeff has never seen a movie.

$\neg \exists m \in \mathcal{M}, M(\text{Jeff}, m)$, which simplifies slightly to $\forall m \in \mathcal{M}, \neg M(\text{Jeff}, m)$.

(b) Jack has seen every movie.

$\forall m \in \mathcal{M}, M(\text{Jack}, m)$

(c) Ellen has seen every movie that Max has seen.

$\forall m \in \mathcal{M}, (\ M(\text{Max}, m) \Rightarrow M(\text{Ellen}, m)\ )$

(d) Everyone has seen at least one movie.

$\forall p \in \mathcal{P}, \exists m \in \mathcal{M}, M(p, m)$

(e) Between the two of them, Jenny and Tom have seen every movie.

$\forall m \in \mathcal{M}, (\ M(\text{Jenny}, m) \vee M(\text{Tom}, m)\ )$

IX. (3) Assuming that the universal set is $\mathcal{U} = \mathbb{R}$, prove the statement $\forall x, \exists y, x < y$.

Let x be a real number. Putting $y = x + 1$, we have $x < x + 1 = y$.

X. (8) Assuming that the universal set is $\mathcal{U} = \mathbb{R}$, prove (if the statement is true) or disprove (if the statement is false) each of the following statements.

1. $\forall x, (x > 1 \Rightarrow x > 0)$

   Proof: Let x be a real number. Assume that $x > 1$. Since $1 > 0$, it follows that $x > 0$.

2. $\exists x, (x > 1 \Rightarrow x > 0)$

   Proof: Put $x = 2$. Then $2 > 1$ and $2 > 0$, so $2 > 1 \Rightarrow 2 > 0$.

3. $\forall x, (x > 0 \Rightarrow x > 1)$

   Counterexample: Putting $x = \frac{1}{2}$, we have $\frac{1}{2} > 0$ but $\frac{1}{2} \leq 1$.

4. $\exists x, (x > 0 \Rightarrow x > 1)$

   Proof: Put $x = 2$. Then $2 > 0$ and $2 > 1$, so $2 > 0 \Rightarrow 2 > 1$.

   A more amusing Proof: Put $x = -1$. Then $-1 > 0$ and $-1 > 1$ are false, so $-1 > 0 \Rightarrow -1 > 1$ is true.

XI. (6) This problem concerns the following statement about integers: "If $3n + 6$ is even, then n is even."

(a) Prove the statement by arguing the contrapositive.

We will argue the contrapositive. Assume that n is odd. Then we can write $n = 2k + 1$ for some integer k. We calculate that $3n + 6 = 6k + 9 = 2(3k + 4) + 1$, so $3n + 6$ is odd.

(b) Prove the statement using proof by contradiction.

Suppose for contradiction that there is an integer n for which $3n + 6$ is even, but n is odd. We can write $n = 2k + 1$ for some integer k, and calculate that $3n + 6 = 6k + 9 = 2(3k + 4) + 1$. Therefore, $3n + 6$ must be odd, contradicting the assumption that it is even.