Mathematics 2513-001 Name (please print) ______________________

Examination II Form B

March 23, 2006

Instructions: Give brief, clear answers. "Prove" means "give an argument". In giving definitions, give the *precise* definition, using logical notation and/or set notation as appropriate.

I. (2) Write the following as an implication: "$b^2 \geq 2$ for at most one b".

$(b^2 \geq 2 \wedge c^2 \geq 2) \Rightarrow b = c$

II. (4) Let $T(p, c)$ be "Person p has traveled to the city c." Write each of the following statements in logical notation, putting in all necessary quantifiers using the sets $\mathcal{P}$ of all people and $\mathcal{C}$ of all destination cities. If your answer involves a negation, simplify as much as possible.

(a) Joan has been to Paris or London.

$T(\text{Joan, Paris}) \vee T(\text{Joan, London})$

(b) Everyone has traveled to at least one city.

$\forall p \in \mathcal{P}, \exists c \in \mathcal{C}, T(p, c),$

(c) No one has traveled to every city.

$\neg\exists p \in \mathcal{P}, \forall c \in \mathcal{C}, T(p, c)$, which simplifies to $\forall p \in \mathcal{P}, \exists c \in \mathcal{C}, \neg T(p, c),$

(d) Any two people have traveled to at least one city in common.

$\forall p \in \mathcal{P}, \forall q \in \mathcal{P}, \exists c \in \mathcal{C}, (T(p, c) \wedge T(q, c))$

III. (10) For the function $H \colon s \to t$ (where s and t are sets), give definitions of the following. As requested in the instructions, give the *precise* definitions, using logical notation and/or set notation as appropriate.

(a) the range of H

$\{H(x) \mid x \in s\}$

(b) the preimage of an element T of t

$\{x \in s \mid H(x) = T\}$

(c) the inverse function H^{-1}, assuming that N is bijective (part of giving the definition of a function is telling its domain and codomain).

$H^{-1} \colon t \to s$ is defined by $H^{-1}(y) = x \Leftrightarrow H(x) = y$

(d) the composition $G \circ H$, assuming that $G \colon t \to u$ (part of giving the definition of a function is telling its domain and codomain).

$G \circ H \colon s \to u$ is defined by $G \circ H(x) = G(H(x))$

(e) $H = K$, where $K \colon u \to v$

$H = K$ when $s = u$, $t = v$, and $\forall x \in s, H(x) = K(x)$

(f) the graph of H

$\{(x, H(x)) \mid x \in s\}$, a subset of $s \times t$.

IV. Let X be an infinite set.
(4) (a) Define what it means to say that X is *countable.*

X is *countable* where there exists a bijective function from $\mathbb{N}$ to X.

(b) Show a function that verifies that $\mathbb{Z}$ is countable.

A bijective function from $\mathbb{N}$ to $\mathbb{Z}$ is given by using the pattern
$1 \mapsto 0$,
$2 \mapsto 1$, $3 \mapsto -1$,
$4 \mapsto 2$, $5 \mapsto -2$,
$6 \mapsto 3$, $7 \mapsto -3$,
$8 \mapsto 4$, $9 \mapsto -4$,
and so on.

V. Write out all elements of $\mathcal{P}(\{1,2\})$ and all elements of $\mathcal{P}(\{1,2\}) \times \{1,2,3\}$.
(4)

$\mathcal{P}(\{1,2\}) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$
$\mathcal{P}(\{1,2\}) \times \{1,2,3\} = \{(\emptyset,1), (\emptyset,2), (\emptyset,3), (\{1\},1), (\{1\},2), (\{1\},3), (\{2\},1), (\{2\},2), (\{2\},3), (\{1,2\},1), (\{1,2\},2), (\{1,2\},3)\}$

VI. Let $f\colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ be the function defined by $f(m,n) = m^2 - n$. Prove that f is surjective.
(4)

Let $k \in \mathbb{Z}$. Then, $f(0,-k) = 0^2 - (-k) = k$.

VII. Let $f\colon X \to Y$ and $g\colon Y \to Z$. Prove that if f and g are injective, then the composition $g \circ f$ is injective.
(4)

Assume that f and g are injective. Let $x_1, x_2 \in X$ and assume that $g \circ f(x_1) = g \circ f(x_2)$. This says that $g(f(x_1)) = g(f(x_2))$. Since g is injective, this implies that $f(x_1) = f(x_2)$. Since f is injective, this implies that $x_1 = x_2$.

VIII. Disprove the following assertion: for all sets A, B, and C, if $A \cap C = B \cap C$, then $A = B$.
(3)

Put $A = \{1,2\}$, $B = \{1,3\}$, and $C = \{1\}$. Then $A \cap C = \{1\}$ and $B \cap C = \{1\}$, but $A \neq B$.

IX. Let $f\colon (0,\pi) \to (0,\infty)$ be the function defined by $f(x) = \csc(x)$, where the cosecant function is as usual
(8) given by $\csc(x) = \dfrac{1}{\sin(x)}$.
(a) Prove that f is not injective.

$$f(\pi/4) = \frac{1}{1/\sqrt{2}} = \sqrt{2} \text{ and } f(3\pi/4) = \frac{1}{1/\sqrt{2}} = \sqrt{2}, \text{ but } \pi/4 \neq 3\pi/4.$$

(b) Prove that f is not surjective.

Consider $1/2$, an element of the codomain $(0,\infty)$. For all $x \in (0,\pi)$, $0 < \sin(x) \leq 1$, so $1 \leq \csc(x)$. Therefore $\csc(x) \neq 1/2$.

(Alternatively, we can use proof by contradiction: Suppose for contradiction that there exists $x \in (0,\pi)$ such that $\csc(x) = 1/2$. Then $1/\sin(x) = 1/2$ so $\sin(x) = 2$, contradicting the fact that $-1 \leq \sin(x) \leq 1$ for all x.)

X. Give an example of a function from $\mathbb{R}$ to $\mathbb{R}$ that is injective but not surjective.
(4)

The exponential function e^x and the inverse tangent function $\tan^{-1}(x)$ are perhaps the most familiar of many possible examples.

XI. Let $A = \mathbb{R}$ and $B = \mathbb{Z}$. Give examples of each of the following.

(3) (a) An element of $A \times B$ that is not in $B \times B$.

$(1/2, 1)$

(b) An element of $B \times A$ that is not in $B \times B$.

$(1, 1/2)$

(c) An element of $A \times A$ that is neither in $A \times B$ nor in $B \times A$.

$(1/2, 1/2)$