Mathematics 2513-001 | Name (please print) ______________________

Examination III Form A

April 27, 2006

Instructions: Give brief, clear answers. "Prove" means "give an argument".

I. (4) Let $f\colon X \to Y$ and $g\colon Y \to Z$. Prove that if f and g are injective, then the composition $g \circ f$ is injective.

II. (4) Prove that if $a \equiv b \mod m$ and $b \equiv c \mod m$, then $a \equiv c \mod m$.

III. (4) Give Euclid's proof that there are infinitely many primes.

IV. (4) State the Fundamental Theorem of Arithmetic.

V. (4) (a) Show that $ac \equiv bc \mod m$ and $c \not\equiv 0 \mod m$ does not always imply that $a \equiv b \mod m$.
(b) Tell without proof a condition (which always holds when m is prime and $c \not\equiv 0 \mod m$) that ensures that $ac \equiv bc \mod m$ does imply that $a \equiv b \mod m$.

VI. (5) Prove that $1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n+1)! - 1$ whenever n is a positive integer.

VII. (5) Let X be the set of all infinite sequences in which each term is one of the letters a, b, or c. Some elements of X are bbbbbbbbbbb$\cdots$, aabbccaabbccaabbcc$\cdots$, and abbabccbbaccbcbacbacbabcabaacbbbbaccbc$\cdots$. Using Cantor's idea, prove that there does not exist any surjective function from $\mathbb{N}$ to X.

VIII. (4) Let Y be the set of all positive fractions (not rational numbers, so $\frac{1}{2}$ and $\frac{2}{4}$ are different fractions). Using Cantor's idea, prove that Y is countable.

IX. (4) Let B be a nonempty set, so that we can choose an element b_0 of B. Prove that there exists a surjective function from $\mathcal{P}(B)$ to B.

X. (4) Let a, b, and c be integers. Using the definition of "divides", prove that if $a|b$ and $b|c$, then $a|c$.

XI. (5) Let Z be an infinite set.
(a) Informally, saying that Z is countable means that it is possible to list the elements of Z. This is not a real definition, since the word "list" is not precise. Give the formal definition of "Z is countable."
(b) Now suppose that Z is set of all infinite sequences in which each term is one of the letters a, or b, and *exactly one* of the terms is b. Some elements of Y are baaaaaaa$\cdots$, aaaaaaaaabaaaaaa$\cdots$, and aaaaaaa$\cdots$aaaabaaaa$\cdots$, where in the last sequence the b appears after exactly $35{,}014{,}227$ a's have appeared. Prove that Z is countable.

XII. (4) Let m and n be two positive integers. Show that if $mn = 360$ and the least common multiple of m and n is 10 times their greatest common divisor, then both m and n are divisible by 6.