Mathematics 2513-001 Name (please print) ______________________

Examination III Form B

April 27, 2006

Instructions: Give brief, clear answers. "Prove" means "give an argument".

I. (4) Let $f\colon X \to Y$ and $g\colon Y \to Z$. Prove that if f and g are injective, then the composition $g \circ f$ is injective.

II. (4) State the Fundamental Theorem of Arithmetic.

III. (4) Prove that if $a \equiv b \mod m$ and $b \equiv c \mod m$, then $a \equiv c \mod m$.

IV. (4) Give Euclid's proof that there are infinitely many primes.

V. (5) Prove that $1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n+1)! - 1$ whenever n is a positive integer.

VI. (4) (a) Show that $ac \equiv bc \mod m$ and $c \not\equiv 0 \mod m$ does not always imply that $a \equiv b \mod m$.
(b) Tell without proof a condition (which always holds when m is prime and $c \not\equiv 0 \mod m$) that ensures that $ac \equiv bc \mod m$ does imply that $a \equiv b \mod m$.

VII. (5) Let X be the set of all infinite sequences in which each term is one of the letters x, y, or z. Some elements of X are yyyyyyyyyyy$\cdots$, xxyyzzxxyyzzxxyyzz$\cdots$, and xyyxyzzyyxzzyzyxzyxzyxyzxyxxzyyyyxzzyz$\cdots$. Using Cantor's idea, prove that there does not exist any surjective function from $\mathbb{N}$ to X.

VIII. (4) Let a, b, and c be integers. Using the definition of "divides", prove that if $a|b$ and $b|c$, then $a|c$.

IX. (4) Let Y be the set of all positive fractions (not rational numbers, so $\frac{1}{2}$ and $\frac{2}{4}$ are different fractions). Using Cantor's idea, prove that Y is countable.

X. (4) Let A be a nonempty set, so that we can choose an element a_0 of A. Prove that there exists a surjective function from $\mathcal{P}(A)$ to A.

XI. (4) Let m and n be two positive integers. Show that if $mn = 360$ and the least common multiple of m and n is 10 times their greatest common divisor, then both m and n are divisible by 6.

XII. (5) Let Z be an infinite set.
(a) Informally, saying that Z is countable means that it is possible to list the elements of Z. This is not a real definition, since the word "list" is not precise. Give the formal definition of "Z is countable."
(b) Now suppose that Z is set of all infinite sequences in which each term is one of the letters a, or b, and *exactly one* of the terms is a. Some elements of Y are abbbbbbb$\cdots$, bbbbbbbbbabbbbbb$\cdots$, and bbbbbbb$\cdots$bbbbabbbb$\cdots$, where in the last sequence the a appears after exactly $35,014,227$ b's have appeared. Prove that Z is countable.