Mathematics 2443-006H Name (please print) ____________________

Examination I

February 14, 2008

Instructions: Give brief, clear answers.

I. Calculate each of the following.
(9)
(a) The directional derivative of $\ln(x^2+y^2)$ at the point $(2,1)$ in the direction toward $(-1,2)$.

$\nabla(\ln(x^2+y^2)) = \dfrac{2x}{x^2+y^2}\vec{\imath} + \dfrac{2y}{x^2+y^2}\vec{\jmath}$ so $\nabla(\ln(x^2+y^2))(2,1) = \dfrac{4}{5}\vec{\imath} + \dfrac{2}{5}\vec{\jmath}$. The vector from $(2,1)$ to $(-1,2)$ is $-3\vec{\imath} + \vec{\jmath}$, which has length $\sqrt{10}$, so a unit vector in that direction is $\vec{u} = \dfrac{-3}{\sqrt{10}}\vec{\imath} + \dfrac{1}{\sqrt{10}}\vec{\jmath}$. The desired rate of change is $\nabla(\ln(x^2+y^2))(2,1)\cdot\vec{u} = \dfrac{-2}{\sqrt{10}}$.

(b) The maximum rate of change of $qe^{-p} - pe^{-q}$ at $(p,q) = (0,0)$, and the direction in which it occurs.

$\nabla(qe^{-p} - pe^{-q}) = (-qe^{-p} - e^{-q})\vec{\imath} + (e^{-p} + pe^{-q})\vec{\jmath}$, so $\nabla(qe^{-p} - pe^{-q})(0,0) = -\vec{\imath} + \vec{\jmath}$. This is the direction of the maximum rate of change, and the rate in that direction is the length $\|-\vec{\imath} + \vec{\jmath}\| = \sqrt{2}$.

(c) An equation for the tangent plane to the level surface of $\sqrt{x^2+y^2+z^2}$ at the point $(1,2,-2)$.

$\nabla(\sqrt{x^2+y^2+z^2}) = \dfrac{x}{\sqrt{x^2+y^2+z^2}}\vec{\imath} + \dfrac{y}{\sqrt{x^2+y^2+z^2}}\vec{\jmath} + \dfrac{z}{\sqrt{x^2+y^2+z^2}}\vec{k}$, so a normal vector to the level surface at the point $(1,2,-2)$ is $\nabla(\sqrt{x^2+y^2+z^2})(1,2,-2) = \dfrac{1}{3}\vec{\imath} + \dfrac{2}{3}\vec{\jmath} - \dfrac{2}{3}\vec{k}$. So an equation of the tangent plane is $\dfrac{1}{3}(x-1) + \dfrac{2}{3}(y-2) - \dfrac{2}{3}(z+2) = 0$, or just $x + 2y - 2z = 9$.
Alternatively, you could just notice that the surface is a sphere, and a radius of the sphere must normal to the sphere, so the position vector $\vec{\imath} + 2\vec{\jmath} - 2\vec{k}$ of $(1,2,-2)$ is normal.

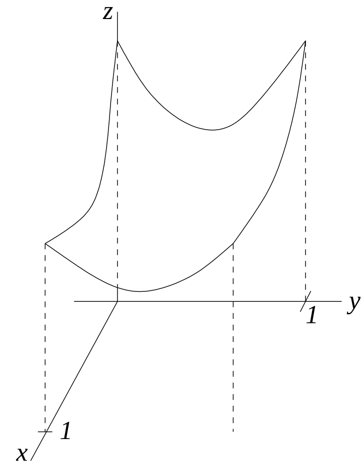
II. Let $f(x,y) = xy - x + 2y$, and let D be the closed triangular region with vertices $(4,0)$, $(0,4)$, and $(0,0)$.
(6) Find the maximum and minimum values of f on the domain D, and where they occur.

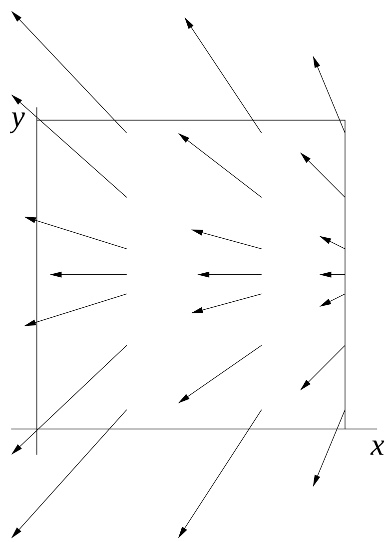
$\dfrac{\partial f}{\partial x} = y - 1$ and $\dfrac{\partial f}{\partial y} = x + 2$, so the only critical point is $(-2,1)$, which does not lie in D.
We now examine f on the three sides:
1. The points on the side from $(0,0)$ to $(4,0)$ are $(x,0)$ for $0 \le x \le 4$, and $f(x,0) = -x$. So the maximum and minimum on this side occur at $(0,0)$ and $(4,0)$.
2. Similarly, $f(0,y) = 2y$, so the maximum and mimumum on the vertical side occur at $(0,0)$ and $(0,4)$.
3. Points on the diagonal side have the form $(x, 4-x)$ for $0 \le x \le 4$, and $f(x,4-x) = -x^2 - x + 8$. This has a critical point when $x = 1/2$, giving the point $(1/2, 7/2)$ as another possibility for an extreme value of f.
Since $f(0,0) = 0$, $f(4,0) = 8$, $f(0,4) = -4$, and $f(1/2,7/2) = 33/4$, the mimimum value of f on D is -4 at $(0,4)$, and the maximum is $33/4$ at $(1/2,7/2)$.

III. Calculate the differential of the function $\sqrt{x^2+y^2}$. Use it to calculate the linear part of the change of
(5) $\sqrt{x^2+y^2}$ going from $(x,y) = (1,1)$ to $(x,y) = (3,2)$.

$d\sqrt{x^2+y^2} = \dfrac{x}{\sqrt{x^2+y^2}}\,dx + \dfrac{y}{\sqrt{x^2+y^2}}\,dy$. For the linear part of the change of $\sqrt{x^2+y^2}$ going from $(x,y) = (1,1)$ to $(x,y) = (3,2)$, we take $(x,y) = (1,1)$, $dx = 2$, and $dy = 1$ to obtain $\dfrac{3}{\sqrt{2}}$.

IV. (5) In an xy-coordinate system, make a reasonable sketch of the gradient of the function whose graph is shown at the right.





V. (4) Partition the domain $D = [0, 10] \times [0, 4]$ into six rectangles, using the partition $\{0, 2, 6, 10\}$ in the x-direction and $\{0, 2, 4\}$ in the y-direction. Using the midpoints as sample points, calculate the Riemann sum of the function $x - 2y$ for this partition.

The midpoints and the areas of the rectangles that contain them are: $(1, 1)$ and 4, $(4, 1)$ and 8, $(8, 1)$ and 8, $(1, 3)$ and 4, $(4, 3)$ and 8, and $(8, 3)$ and 8. So the Riemann sum is

$$f(1,1)\times 4+f(4,1)\times 8+f(8,1)\times 8+f(1,3)\times 4+f(4,3)\times 8+f(8,3)\times 8 = -4+16+48-20-16+16 = 40\ .$$

VI. Let $x = e^u \sin(t)$, $y = e^u \cos(t)$, and $z = f(x, y)$.
(7)

1. Calculate $\dfrac{\partial x}{\partial t}$ and $\dfrac{\partial y}{\partial t}$.

   $\dfrac{\partial x}{\partial t} = e^u \cos(t) = y$ and $\dfrac{\partial y}{\partial t} = -e^u \sin(t) = -x$.

2. Calculate $\dfrac{\partial z}{\partial t}$ and express it purely in terms of x, y, $\dfrac{\partial z}{\partial x}$, and $\dfrac{\partial z}{\partial y}$.

   $\dfrac{\partial z}{\partial t} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial t} + \dfrac{\partial z}{\partial x}\dfrac{\partial y}{\partial t} = \dfrac{\partial z}{\partial x} y - \dfrac{\partial z}{\partial y} x.$

3. Calculate $\dfrac{\partial}{\partial t}\left(\dfrac{\partial z}{\partial x} x\right)$ and express it purely in terms of x and y and partial derivatives of z.

   Applying the Chain Rule, we have

$$\frac{\partial}{\partial t}\left(\frac{\partial z}{\partial x} x\right) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x} x\right)\frac{\partial x}{\partial t} + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x} x\right)\frac{\partial y}{\partial t}$$
$$= \left(\frac{\partial^2 z}{\partial x^2} x + \frac{\partial z}{\partial x}\right) y + \left(\frac{\partial^2 z}{\partial x\,\partial y} x\right)(-x) = \frac{\partial^2 z}{\partial x^2} xy - \frac{\partial^2 z}{\partial x\,\partial y} x^2 + \frac{\partial z}{\partial x} y\ .$$

   Alternatively, one could try to use the product rule and Clairaut's Theorem:

$$\frac{\partial}{\partial t}\left(\frac{\partial z}{\partial x} x\right) = \frac{\partial^2 z}{\partial t\,\partial x} x + \frac{\partial z}{\partial x}\,\frac{\partial x}{\partial t} = x\,\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial t}\right) + \frac{\partial z}{\partial x}\,\frac{\partial x}{\partial t} = x\,\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x} y - \frac{\partial z}{\partial y} x\right) + \frac{\partial z}{\partial x}\,y$$
$$= x\left(\frac{\partial^2 z}{\partial x^2} y - \frac{\partial^2 z}{\partial x\,\partial y} x - \frac{\partial z}{\partial y}\right) + \frac{\partial z}{\partial x}\,y = \frac{\partial^2 z}{\partial x^2} xy - \frac{\partial^2 z}{\partial x\,\partial y} x^2 + \frac{\partial z}{\partial x} y - \frac{\partial z}{\partial y} x\ .$$

   Note that this gives a different answer, so one of the two calculations is incorrect. It turns out that the second is incorrect, for a subtle reason (I still gave this answer full credit). The reason is that Clairaut's Theorem is not being correctly: x and t are not independent variables, so it is not really the setup of Clairaut's Theorem. Here is a simple example from 1-variable calculus that shows that such a calculation does not work:

$$\frac{d}{dt}\,\frac{d}{d(t^2)}\,(t^2) = \frac{d}{dt}\,(1) = 0$$

   but (assuming that $t > 0$):

$$\frac{d}{d(t^2)}\,\frac{d}{dt}\,(t^2) = \frac{d}{d(t^2)}\,(2t) = \frac{d}{d(t^2)}\,(2\sqrt{t^2}) = 2\cdot\frac{1}{2\sqrt{t^2}} = \frac{1}{t}$$

   There is simply no reason that these calculations need to give the same answer.

VII. (4) Use implicit differentiation to calculate $\dfrac{\partial R}{\partial R_2}$ if

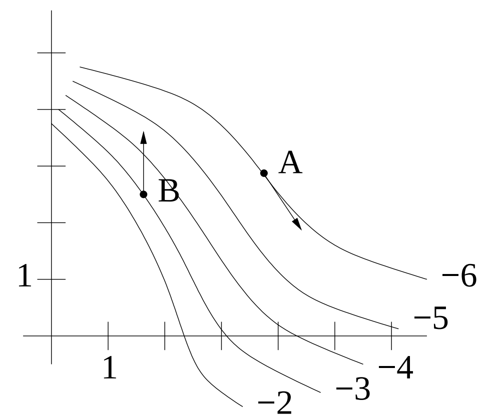
$$\frac{1}{\sin(R)} = \frac{1}{\sin(R_1R_2)} + \frac{1}{\sin(R_1R_3)}\ .$$

$$-\frac{1}{\sin^2(R)}\cos(R)\frac{\partial R}{\partial R_2} = -\frac{1}{\sin^2(R_1R_2)}\cos(R_1R_2)\,R_1 + 0$$
$$\frac{\partial R}{\partial R_2} = \frac{\sin^2(R)\,\cos(R_1R_2)\,R_1}{\cos(R)\,\sin^2(R_1R_2)} = \frac{R_1\,\sin(R)\,\tan(R)}{\sin(R_1R_2)\,\tan(R_1R_2)}$$

or

$$\csc(R) = \csc(R_1R_2) + \csc(R_1R_3)$$
$$-\csc(R)\cot(R)\frac{\partial R}{\partial R_2} = -\csc(R_1R_2)\cot(R_1R_2)R_1 + 0$$
$$\frac{\partial R}{\partial R_2} = \sin(R_1)\tan(R_1)\csc(R_1R_2)\cot(R_1R_2)R_1 = \frac{\sin^2(R)\,\cos(R_1R_2)\,R_1}{\cos(R)\,\sin^2(R_1R_2)} = \frac{R_1\,\sin(R)\,\tan(R)}{\sin(R_1R_2)\,\tan(R_1R_2)}$$

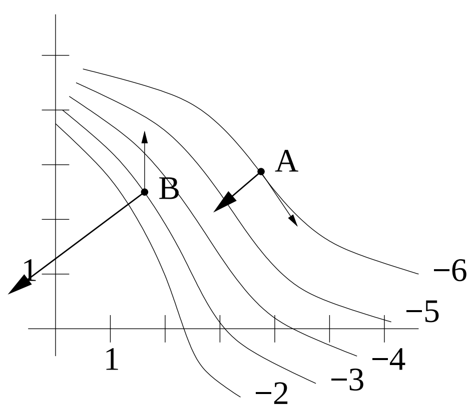
VIII. (6) In the xy-coordinate system to the right, the level curves $f(x,y) = c$ of a function are shown for $c = -2, -3, -4, -5$, and -6, along with two points A and B, and a unit vector at each of the points A and B.



1. Sketch reasonable possibilities for ∇f at the points A and B.
2. Make a reasonable guess of the rate of change of f at A in the direction of the vector shown there.
3. Make a reasonable guess of the rate of change of f at B in the direction of the vector shown there.

The gradient at A is perpendicular to the level curve and points in the direction of larger values. It should have length around 1, since the distance from the -6 level curve to the level -5 level curve is around 1.

The gradient at B should have length around 3, since the distance from the -6 level curve to the level -5 level curve is around $1/3$ (traveling at unit speed in that direction, one is crossing unit level curves at around three per unit time).



A reasonable guess for the rate of change at A is 0, since the vector appears to be tangent to the level curve. For B, the projection of the gradient to the direction of the other vector would have length around 2, but point in the opposite direction. So the rate of change $\nabla f \cdot \vec{u}$ should be around -2.

IX. (6) A unit vector $\vec{u}$ in 3-dimensional space can be written as $a\,\vec{\imath}+b\,\vec{\jmath}+c\,\vec{k}$ where a, b, and c are numbers satisfying $a^2+b^2+c^2=1$. Let $f(x,y,z)$ be a function on xyz-space.

(i) Write parametric equations for the straight line through the point (x_0,y_0,z_0) with direction vector $\vec{u}=a\,\vec{\imath}+b\,\vec{\jmath}+c\,\vec{k}$. (That is, find functions $x(t)$, $y(t)$, and $z(t)$ so that $x=x(t)$, $y=y(t)$, and $z=z(t)$ are parametric equations for this line.)

Using the standard formula for the line through (x_0,y_0,z_0) with direction vector $\vec{u}=a\,\vec{\imath}+b\,\vec{\jmath}+c\,\vec{k}$ gives $x=x_0+at$, $y=y_0+bt$, and $z=z_0+ct$.

(ii) Put your explicit functions $x(t)$, $y(t)$, and $z(t)$ into the expression $f(x(t),y(t),z(t)))$ to find an expression for the values of f along the straight line. Use the Chain Rule to calculate $\frac{d}{dt}(f(x(t),y(t),z(t)))$.

$$\frac{d}{dt}(x_0+at,y_0+bt,z_0+ct)$$
$$=\frac{\partial f}{\partial x}(x_0+at,y_0+bt,z_0+ct)\frac{d(x_0+at)}{dt}$$
$$+\frac{\partial f}{\partial y}(x_0+at,y_0+bt,z_0+ct)\frac{d(y_0+bt)}{dt}+\frac{\partial f}{\partial z}(x_0+at,y_0+bt,z_0+ct)\frac{d(z_0+ct)}{dt}$$
$$=\frac{\partial f}{\partial x}(x_0+at,y_0+bt,z_0+ct)a+\frac{\partial f}{\partial y}(x_0+at,y_0+bt,z_0+ct)b+\frac{\partial f}{\partial z}(x_0+at,y_0+bt,z_0+ct)c\ .$$

(iii) Find the value of your expression for $\frac{d}{dt}(f(x(t),y(t),z(t)))$ when $t=0$ and verify that it equals $\nabla f(x_0,y_0,z_0)\cdot\vec{u}$.

$$\frac{\partial f}{\partial x}(x_0,y_0,z_0)a+\frac{\partial f}{\partial y}(x_0,y_0,z_0)b+\frac{\partial f}{\partial z}(x_0,y_0,z_0)c$$
$$=\left(\frac{\partial f}{\partial x}(x_0,y_0,z_0)\,\vec{\imath}+\frac{\partial f}{\partial y}(x_0,y_0,z_0)\,\vec{\jmath}+\frac{\partial f}{\partial z}(x_0,y_0,z_0)\,\vec{k}\right)\cdot(a\,\vec{\imath}+b\,\vec{\jmath}+c\,\vec{k})=\nabla f(x_0,y_0,z_0)\cdot\vec{u}\ .$$