Mathematics 2443-003 Name (please print) ____________________

Examination III

April 24, 2008

Instructions: Give brief answers, but clearly indicate your reasoning.

$x = \rho\cos(\theta)\sin(\phi)$, $y = \rho\sin(\theta)\sin(\phi)$, $z = \rho\cos(\phi)$, $dV = \rho^2\sin(\phi)\,d\rho\,d\phi\,d\theta$, $\vec{r}_\phi \times \vec{r}_\theta = a\,\sin(\phi)(x\vec{\imath} + y\vec{\jmath} + z\vec{k})$,
$\|\,\vec{r}_\phi \times \vec{r}_\theta\,\| = a^2\,\sin(\phi)$

$dS = \sqrt{1 + g_x^2 + g_y^2}\;dD$

$dS = \|\,\vec{r}_u \times \vec{r}_v\,\|\;dD$

$\iint_S\;\vec{F}\cdot d\vec{S} = \iint_S\;\vec{F}\cdot\vec{n}\;dS$

$\iint_S\;(P\,\vec{\imath} + Q\,\vec{\jmath} + R\,\vec{k})\cdot d\vec{S} = \iint_D\;-P\,g_x - Q\,g_y + R\;dD$

$\iint_S\;\vec{F}\cdot d\vec{S} = \iint_D\;\vec{F}\cdot(\vec{r}_u \times \vec{r}_v)\;dD$

I. (6) For the following line integrals: write a definite integral, in terms of the specified parameter, whose value equals the value of the line integral, but *do not* evaluate the definite integral.

1. $\displaystyle\int_C xy^2\,ds$, where C is parameterized by $x = -t^2$, $y = t^3$ for $1 \le t \le 2$.

2. $\displaystyle\int_C (xy^2\vec{\imath})\cdot d\vec{r}$, where C is parameterized by $x = -t^2$, $y = t^3$ for $1 \le t \le 2$.

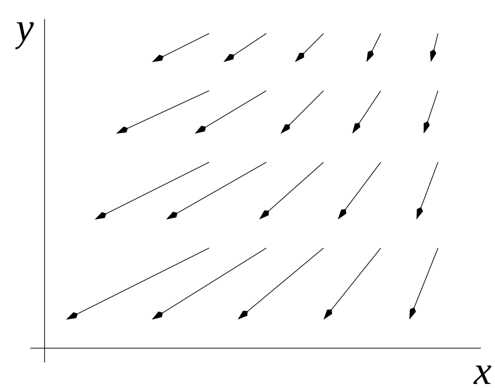
II. (6) Let r and θ be the usual polar coordinates on the plane. Calculate $\dfrac{\partial\theta}{\partial x}$ as follows:

(i) Use implicit differentiation starting from $r^2 = x^2 + y^2$ to calculate that $\dfrac{\partial r}{\partial x} = \dfrac{x}{r}$.

(ii) Starting from $x = r\cos(\theta)$, obtain an expression for $\dfrac{\partial\theta}{\partial x}$ in terms of x and y.

III. (4) Find a normal vector to the surface given by the parameterization $x = u^2$, $y = 2u\sin(v)$, $z = u\cos(v)$ at the point corresponding to $(u, v) = (2, \pi/4)$.

IV. (6) The figure to the right shows a vector field $\vec{F} = P\vec{\imath} + Q\vec{\jmath}$. Based on the probable behavior of the vector field, tell whether each of the following is zero, positive, negative, or cannot reasonably be determined from the information given: $\dfrac{\partial P}{\partial x}$, $\dfrac{\partial Q}{\partial x}$, $\dfrac{\partial P}{\partial y}$, $\dfrac{\partial Q}{\partial y}$, $\operatorname{div}(\vec{F})$, $\operatorname{curl}(\vec{F})\cdot\vec{k}$.



V. (5) State the Fundamental Theorem for Line Integrals, and use it to evaluate $\displaystyle\int_C (x^2\vec{\imath} + y\vec{\jmath})\cdot d\vec{r}$, where C is a path from $(1, 0)$ to $(2, 2)$.

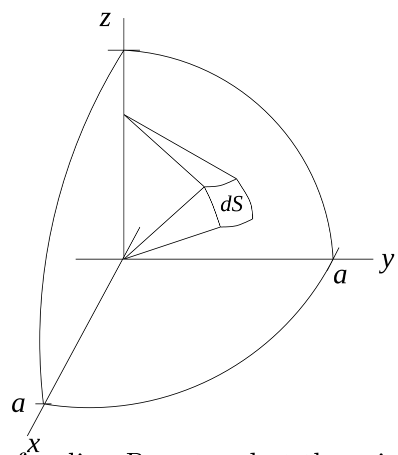
VI. (5) Calculate $\displaystyle\int_C xe^{-x}\,dx + (x^3 + 3xy^2)\,dy$, where C is the unit circle with the clockwise orientation.

VII. (12) Set up definite integrals whose values equal the values of the following surface integrals. Supply limits of integration, but *do not* carry out the evaluation of the definite integrals.

1. $\displaystyle\iint_S y\,dS$ where S is the part of the paraboloid $y = x^2 + z^2$ that lies inside the cylinder $x^2 + z^2 = 4$. Express the definite integral in *polar coordinates* in the xz-plane, including specifying the limits of integration, but do not evaluate.

2. $\displaystyle\iint_S (x\vec{\imath} + x\vec{\jmath} + 2z\vec{k}) \cdot d\vec{S}$, where S is the portion of the sphere $x^2 + y^2 + z^2 = 4$ with $0 \le \phi \le 3\pi/4$, and with respect to the outward normal. Make use of the formulas given at the start of the exam; do not derive expressions for $\vec{r}_\phi$ and $\vec{r}_\theta$.

VIII. (3) Let S be the sphere of radius a with center at the origin. The differential of surface area on S can be expressed in terms of $d\phi$ and $d\theta$. Using the picture shown to the right, explain why dS appears to be $a^2 \sin(\phi)\, d\phi\, d\theta$.



IX. (8) Let $\vec{F}(x, y)$ be the vector field $\dfrac{-y}{x^2+y^2}\vec{\imath} + \dfrac{x}{x^2+y^2}\vec{\jmath}$. Let C_R be the circle of radius R centered at the origin of the xy-plane.

(i) Evaluate $\displaystyle\int_{C_R} \vec{F} \cdot d\vec{r}$ by direct calculation using a parameterization of C_R.

(ii) Let C be any simple (no self crossings) loop that encloses the origin. Give C the positive orientation. Let R be a number so large that C is entirely contained inside C_R, and let D be the region lying between C_R and C, so that the oriented boundary of D is $C_R + (-C)$.

(a) Draw a sketch of a typical C and C_R, showing the region D.

(b) Use Green's Theorem, which applies to the region D as long as we used its oriented boundary $C_R + (-C)$, to calculate that $\displaystyle\int_{C_R+(-C)} \vec{F} \cdot d\vec{r} = 0$.

(c) What is the numerical value of $\displaystyle\int_C \vec{F} \cdot d\vec{r}$? Why?