Mathematics 2443-003 Name (please print) ______________________

Examination III

April 24, 2008

Instructions: Give brief answers, but clearly indicate your reasoning.

$x = \rho\cos(\theta)\sin(\phi)$, $y = \rho\sin(\theta)\sin(\phi)$, $z = \rho\cos(\phi)$, $dV = \rho^2\sin(\phi)\,d\rho\,d\phi\,d\theta$, $\vec{r}_\phi \times \vec{r}_\theta = a\ \sin(\phi)(x\vec{\imath} + y\vec{\jmath} + z\vec{k})$,
$\|\,\vec{r}_\phi \times \vec{r}_\theta\,\| = a^2\ \sin(\phi)$
$dS = \sqrt{1 + g_x^2 + g_y^2}\ dD$
$dS = \|\,\vec{r}_u \times \vec{r}_v\,\|\ dD$
$\iint_S\ \vec{F}\cdot d\vec{S} = \iint_S\ \vec{F}\cdot\vec{n}\ dS$
$\iint_S\ (P\,\vec{\imath} + Q\,\vec{\jmath} + R\,\vec{k})\cdot d\vec{S} = \iint_D\ -P\,g_x - Q\,g_y + R\ dD$
$\iint_S\ \vec{F}\cdot d\vec{S} = \iint_D\ \vec{F}\cdot(\vec{r}_u \times \vec{r}_v)\ dD$

I. For the following line integrals: write a definite integral, in terms of the specified parameter, whose value equals the value of the line integral, but *do not* evaluate the definite integral.
(6)

1. $\displaystyle\int_C xy^2\,ds$, where C is parameterized by $x = -t^2$, $y = t^3$ for $1 \le t \le 2$.

$$ds = \sqrt{(-2t)^2 + (3t^2)^2}\,dt = \sqrt{4t^2 + 9t^4}\,dt, \text{ so}$$
$$\int_C xy^2\,ds = \int_1^2 (-t^2)(t^3)^2\sqrt{4t^2 + 9t^4}\,dt = \int_1^2 -t^9\sqrt{4 + 9t^2}\,dt.$$

2. $\displaystyle\int_C (xy^2\vec{\imath})\cdot d\vec{r}$, where C is parameterized by $x = -t^2$, $y = t^3$ for $1 \le t \le 2$.

$$\vec{r}\,'(t) = -2t\,\vec{\imath} + 3t^2\vec{\jmath}, \text{ so } \int_C (xy^2\vec{\imath})\cdot d\vec{r} = \int_1^2 ((-t^2)(t^3)^2\vec{\imath})\cdot(-2t\,\vec{\imath} + 3t^2\vec{\jmath})\,dt = \int_1^2 2t^9\,dt.$$

II. Let r and θ be the usual polar coordinates on the plane. Calculate $\dfrac{\partial\theta}{\partial x}$ as follows:
(6)

(i) Use implicit differentiation starting from $r^2 = x^2 + y^2$ to calculate that $\dfrac{\partial r}{\partial x} = \dfrac{x}{r}$.

$$r^2 = x^2 + y^2$$
$$2r\frac{\partial r}{\partial x} = 2x$$
$$\frac{\partial r}{\partial x} = \frac{x}{r}$$

(ii) Starting from $x = r\cos(\theta)$, obtain an expression for $\dfrac{\partial\theta}{\partial x}$ in terms of x and y.

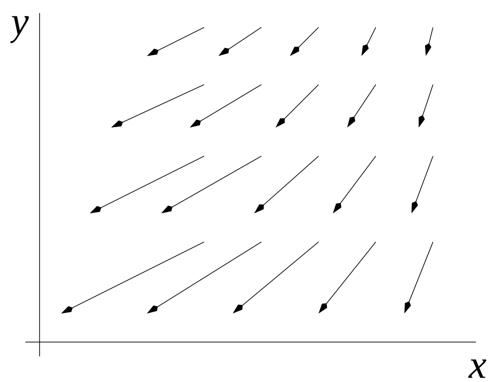
$$x = r\cos(\theta)$$
$$1 = \frac{\partial r}{\partial x}\cos(\theta) - r\sin(\theta)\frac{\partial\theta}{\partial x} = \frac{x}{r}\cos(\theta) - y\frac{\partial\theta}{\partial x}$$
$$y\frac{\partial\theta}{\partial x} = \frac{x}{r}\cos(\theta) - 1 = \frac{x}{r^2}r\cos(\theta) - 1 = \frac{x^2}{r^2} - \frac{r^2}{r^2} = \frac{x^2 - r^2}{r^2} = \frac{-y^2}{r^2}$$
$$\frac{\partial\theta}{\partial x} = \frac{-y}{x^2 + y^2}$$

III. (4) Find a normal vector to the surface given by the parameterization $x = u^2$, $y = 2u\sin(v)$, $z = u\cos(v)$ at the point corresponding to $(u, v) = (2, \pi/4)$.

$\vec{r}_u = 2u\,\vec{\imath} + 2\sin(v)\vec{\jmath} + \cos(v)\vec{k}$, $\vec{r}_v = 2u\cos(v)\vec{\jmath} - u\sin(v)\vec{k}$, so
$\vec{r}_u(2, \pi/4) = 4\vec{\imath} + \sqrt{2}\vec{\jmath} + (1/\sqrt{2})\vec{k}$ and $\vec{r}_v(2, \pi/4) = 2\sqrt{2}\vec{\jmath} - \sqrt{2}\vec{k}$, and a normal vector is

$$\vec{r}_u(2, \pi/4) \times \vec{r}_v(2, \pi/4) = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 4 & \sqrt{2} & 1/\sqrt{2} \\ 0 & 2\sqrt{2} & \sqrt{2} \end{vmatrix} = -4\,\vec{\imath} + 4\sqrt{2}\,\vec{\jmath} + 8\sqrt{2}\,\vec{k}\ .$$

IV. (6) The figure to the right shows a vector field $\vec{F} = P\vec{\imath} + Q\vec{\jmath}$. Based on the probable behavior of the vector field, tell whether each of the following is zero, positive, negative, or cannot reasonably be determined from the information given: $\dfrac{\partial P}{\partial x}$, $\dfrac{\partial Q}{\partial x}$, $\dfrac{\partial P}{\partial y}$, $\dfrac{\partial Q}{\partial y}$, $\mathrm{div}(\vec{F})$, $\mathrm{curl}(\vec{F}) \cdot \vec{k}$.



$\dfrac{\partial P}{\partial x}$ is positive, $\dfrac{\partial Q}{\partial x}$ is zero, $\dfrac{\partial P}{\partial y}$ is positive, $\dfrac{\partial Q}{\partial y}$ is positive, $\mathrm{div}(\vec{F}) = \dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y}$ is positive, $\mathrm{curl}(\vec{F}) \cdot \vec{k} = \dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}$ is negative. The last two can also be seen geometrically: the flow is expanding so $\mathrm{div}(\vec{F})$ is positive, and the right-hand rule shows that $\mathrm{curl}(\vec{F})$ points into the page, so its dot product with $\vec{k}$ must be negative.

V. (5) State the Fundamental Theorem for Line Integrals, and use it to evaluate $\displaystyle\int_C (x^2\vec{\imath} + y\vec{\jmath}) \cdot d\vec{r}$, where C is a path from $(1, 0)$ to $(2, 2)$.

Let C be a path from p to q. Then for any function f, $\displaystyle\int_C \nabla f \cdot d\vec{r} = f(q) - f(p)$. In the case of the vector field $x^2\vec{\imath} + y\vec{\jmath}$, it is easy to calculate (or just by inspection see) that $x^2\vec{\imath} + y\vec{\jmath} = \nabla(x^3/3 + y^2/2)$, so $\displaystyle\int_C (x^2\vec{\imath} + y\vec{\jmath}) \cdot d\vec{r} = (2^3/3 + 2^2/2) - (1^3/3 - 0) = 13/3$.

VI. (5) Calculate $\displaystyle\int_C xe^{-x}\,dx + (x^3 + 3xy^2)\,dy$, where C is the unit circle with the clockwise orientation.

This vector field is defined at all points of the unit disk, so letting D be the unit disk we may apply Green's Theorem to find that $\displaystyle\int_C (xe^{-x}\,dx + (x^3 + 3xy^2)\,dy = -\iint_D \frac{\partial}{\partial x}(x^3 + 3xy^2) - \frac{\partial}{\partial y}(xe^{-x})\,dA = -\iint_D \frac{\partial}{\partial x}(x^3 + 3xy^2) - \frac{\partial}{\partial y}(xe^{-x})\ dA = -\iint_D 3x^2 + 3y^2\ dA = -\int_0^{2\pi}\int_0^1 3r^3\,dr\,d\theta = -\frac{3\pi}{2}$. The minus sign is because C is oriented clockwise, so sees the interior of the unit disk on its right.

VII. (12) Set up definite integrals whose values equal the values of the following surface integrals. Supply limits of integration, but *do not* carry out the evaluation of the definite integrals.

1. $\iint_S y\,dS$ where S is the part of the paraboloid $y = x^2 + z^2$ that lies inside the cylinder $x^2 + z^2 = 4$. Express the definite integral in *polar coordinates* in the xz-plane, including specifying the limits of integration, but do not evaluate.

   Regarding y as a function of x and z, with domain the disk R of radius 2 in the xz-plane, we calculate $dS = \sqrt{1 + \left(\frac{\partial y}{\partial x}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2}\,dR = \sqrt{1 + 4x^2 + 4z^2}\,dR$. Therefore we have
   $\iint_S y\,dS = \iint_R (x^2 + z^2)\sqrt{1 + 4x^2 + 4z^2}\,dR = \int_0^{2\pi}\int_0^2 r^3\sqrt{1 + 4r^2}\,dr\,d\theta.$

2. $\iint_S (x\vec{\imath} + x\vec{\jmath} + 2z\vec{k}) \cdot d\vec{S}$, where S is the portion of the sphere $x^2 + y^2 + z^2 = 4$ with $0 \le \phi \le 3\pi/4$, and with respect to the outward normal. Make use of the formulas given at the start of the exam; do not derive expressions for $\vec{r}_\phi$ and $\vec{r}_\theta$.

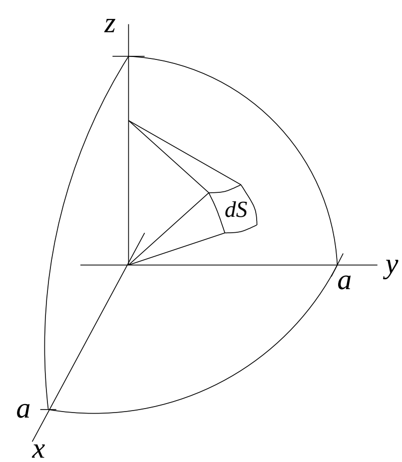
   We have

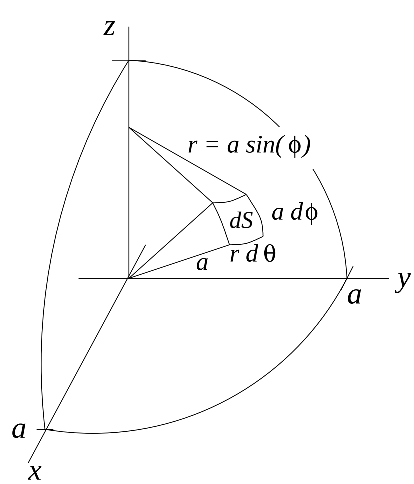
$$\iint_S (x\vec{\imath} + x\vec{\jmath} + 2z\vec{k}) \cdot d\vec{S} = \iint_S (x\vec{\imath} + x\vec{\jmath} + 2z\vec{k}) \cdot (\vec{r}_\phi \times \vec{r}_\theta)\,dS$$

$$= \iint_S (x\vec{\imath} + x\vec{\jmath} + 2z\vec{k}) \cdot (2\,\sin(\phi)(x\vec{\imath} + y\vec{\jmath} + z\vec{k})\,dS = \iint_S 2\sin(\phi)(2x + xy + 2z^2)\,dS$$

$$= \int_0^{2\pi}\int_0^{3\pi/4} 8\sin^3(\phi)\cos(\theta) + 8\sin^3(\phi)\cos(\theta)\sin(\theta) + 16\sin(\phi)\cos^2(\phi)\,d\phi\,d\theta\ .$$

VIII. (3) Let S be the sphere of radius a with center at the origin. The differential of surface area on S can be expressed in terms of $d\phi$ and $d\theta$. Using the picture shown to the right, explain why dS appears to be $a^2\sin(\phi)\,d\phi\,d\theta$.





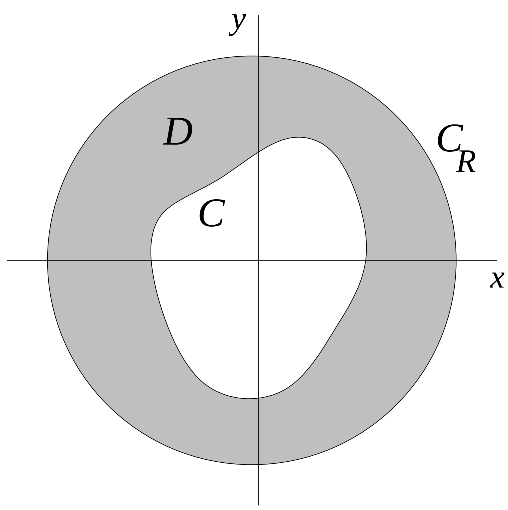
IX. (8) Let $\vec{F}(x,y)$ be the vector field $\dfrac{-y}{x^2+y^2}\vec{\imath}+\dfrac{x}{x^2+y^2}\vec{\jmath}$. Let C_R be the circle of radius R centered at the origin of the xy-plane.

(i) Evaluate $\displaystyle\int_{C_R}\vec{F}\cdot d\vec{r}$ by direct calculation using a parameterization of C_R.

$x=R\cos(t)$, $y=R\sin(t)$, $0\le t\le 2\pi$, $\vec{r}'(t)=-R\sin(t)\vec{\imath}+R\cos(t)\vec{\jmath}$, so
$\displaystyle\int_{C_R}\left(\frac{-y}{x^2+y^2}\vec{\imath}+\frac{x}{x^2+y^2}\vec{\jmath}\right)\cdot d\vec{r}=\int_0^{2\pi}\left(\frac{-R\sin(t)}{R^2}\vec{\imath}+\frac{R\cos(t)}{R^2}\vec{\jmath}\right)\cdot(-R\sin(t)\vec{\imath}+R\cos(t)\vec{\jmath})\,dt=\int_0^{2\pi}dt=$
2π .

(ii) Let C be any simple (no self crossings) loop that encloses the origin. Give C the positive orientation. Let R be a number so large that C is entirely contained inside C_R, and let D be the region lying between C_R and C, so that the oriented boundary of D is $C_R+(-C)$.

(a) Draw a sketch of a typical C and C_R, showing the region D.



(b) Use Green's Theorem, which applies to the region D as long as we used its oriented boundary $C_R+(-C)$, to calculate that $\displaystyle\int_{C_R+(-C)}\vec{F}\cdot d\vec{r}=0$.

First we calculate that $\dfrac{\partial Q}{\partial x}=\dfrac{\partial}{\partial x}\left(\dfrac{x}{x^2+y^2}\right)=\dfrac{(x^2+y^2)-x(2x)}{(x^2+y^2)^2}=\dfrac{y^2-x^2}{(x^2+y^2)^2}$, and similarly $\dfrac{\partial P}{\partial y}=\dfrac{y^2-x^2}{(x^2+y^2)^2}$, so Green's Theorem gives $\displaystyle\int_{C_R+(-C)}\vec{F}\cdot d\vec{r}=\iint_D 0\,dA=0$.

(c) What is the numerical value of $\displaystyle\int_C\vec{F}\cdot d\vec{r}$? Why?

We have $\displaystyle 0=\int_{C_R+(-C)}\vec{F}\cdot d\vec{r}=\int_{C_R}\vec{F}\cdot d\vec{r}+\int_{-C}\vec{F}\cdot d\vec{r}=\int_{C_R}\vec{F}\cdot d\vec{r}-\int_C\vec{F}\cdot d\vec{r}$, and therefore $\displaystyle\int_C\vec{F}\cdot d\vec{r}=\int_{C_R}\vec{F}\cdot d\vec{r}=2\pi$.