Mathematics 3333-001 Name (please print) ____________________

Final Examination

May 13, 2009

Instructions: Give concise answers, but clearly indicate your reasoning.

I. (9) (a) Suppose that $A_1, A_2, \ldots, A_k$ are nonsingular $n \times n$ matrices. Explain why the product $A_1 A_2 \cdots A_k$ is nonsingular.

(b) Give an example of nonzero 2×2 matrices A, B, and C for which $AB = AC$ but $B \neq C$.

(c) Show that if A, B, and C are 2×2 matrices with $AB = AC$, and A is nonsingular, then $B = C$.

II. (3) As you know, a *matrix transformation* from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a function of the form $F(X) = AX$, where A is an $m \times n$ matrix and a point X in $\mathbb{R}^n$ is regarded as an $n \times 1$ column vector. Verify (using properties of matrix addition, matrix multiplication, and scalar multiplication) that any matrix function F must be linear.

III. (7) Let $L\colon V \to W$ be a linear transformation.

(a) Define the *kernel* of L. Verify that it is a subspace of V.

(b) Define the *range* of L (it is a subspace of W, but you do not need to verify this).

IV. (4) Let V be a vector space and let $S = \{v_1, \ldots, v_k\}$ be a subset of V. Verify that $\operatorname{span}(S)$ is a subspace of V.

V. (6) A certain matrix A has 15 rows and 20 columns.

(a) What rank must A have in order that its null space have dimension 8? Explain why.

(b) What nullity must A have in order that the nonhomogeneous system $AX = B$ have at least one solution for every choice of B? Explain why.

VI. (5) A certain matrix transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ is given by multiplication by a 3×3 matrix A, of nullity 1. Draw our standard picture of two 3-dimensional spaces, representing the domain and codomain of the matrix transformation, showing a possible null space, row space, and column space for A (label which is which, in your picture).

VII. (4) Let A be the 4×4 matrix $\begin{bmatrix} t & 0 & 0 & 1 \\ 0 & 0 & t & 0 \\ 0 & t & 0 & 0 \\ 1 & 0 & 0 & t \end{bmatrix}$, which depends on the value of the variable t. Use the *cofactor expansion* method, expanding *across the second row,* to calculate the determinant of A.

VIII. (5) Let $A = \begin{bmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ t & 0 & 1 \end{bmatrix}$, for which $\det(A) = 1 - t^2$. Assuming that $t \neq 1$ and $t \neq -1$, so that A is nonsingular, use the row operation method to compute A^{-1}. (Hint: Start with the elementary row operation $R_3 - tR_1 \to R_3$.)

IX. (5) Let u and v be vectors in an inner product space V.

(a) Use properties of the inner product to determine how $\| u+v \|^2$ is related to $\| u \|^2 + \| v \|^2$. (Hint: $\| u+v \|^2 = (u+v, u+v)$.)

(b) Deduce that if u and v are orthogonal for the inner product, then $\| u+v \|^2 = \| u \|^2 + \| v \|^2$.

X. (6) Let $S = \{v_1, \ldots, v_n\}$ be a set of vectors in a vector space V.

(a) Define what it means to say that S is *linearly independent.*

(b) Show that if S is linearly independent, and

$$a_1v_1 + a_2v_2 + \cdots + a_nv_n = b_1v_1 + b_2v_2 + \cdots + b_nv_n \ ,$$

then $a_i = b_i$ for all i. (Hint: subtract $b_1v_1 + b_2v_2 + \cdots + b_nv_n$ from both sides of the equation.)

XI. (4) Find the characteristic polynomial of the matrix $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 3 & 2 \end{bmatrix}$.

XII. (6) The eigenvalues of the matrix $A = \begin{bmatrix} -2 & -2 & 3 \\ 0 & 3 & -2 \\ 0 & -1 & 2 \end{bmatrix}$ are -2, 1, and 4. Find an eigenvector associated to -2, and an eigenvector associated to 1.

XIII. (5) Find a basis for the eigenspace associated with $\lambda = 3$ for the matrix $A = \begin{bmatrix} 3 & 0 & 0 \\ -2 & 3 & -2 \\ 2 & 0 & 5 \end{bmatrix}$.

XIV. (6) For the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 2 & 1 & 2 \end{bmatrix}$ the eigenvalues are -1, 1, and 4, and associated eigenvectors are $\begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 1 \\ -6 \\ 4 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ (you do not need to calculate these or check them).

(a) Write down a diagonal matrix D that is similar to A.

(b) Write down a matrix P so that $P^{-1}AP = D$.