Mathematics 3333-001 Name (please print) ______________________

Examination III

April 28, 2009

Instructions: Give concise answers, but clearly indicate your reasoning.

I. (8) Let A be the matrix $\begin{bmatrix} 2 & 4 & -2 & 0 \\ 1 & 3 & -1 & -3 \\ 2 & 5 & -2 & -3 \end{bmatrix}$.

(a) Find a basis for the row space of A.

(b) Find a basis for the column space of A.

II. (6) A certain 14×10 matrix A has rank 8.

(a) What is the dimension of the solution space of the homogeneous system $AX = 0$?

(b) Can one solve the linear system $AX = B$ for all choices of B? Why or why not?

III. (7) Let V be a vector space with an inner product $(\_\,,\_)$.

(a) Let w_0 be a fixed vector in V. Show that the set of vectors orthogonal to w_0 is a subspace of V.

(b) Define what it means to say that a set of vectors $S = \{v_1, \ldots, v_n\}$ in V is *orthogonal.*

(c) Define what it means to say that a set of vectors $S = \{v_1, \ldots, v_n\}$ in V is *orthonormal.*

IV. (3) By counting the number of inversions in the permuation 48253176 of eight elements, determine whether this permutation is even or odd.

V. (3) Use the *row operation* method to calculate the determinant $\begin{vmatrix} 3 & 4 & 2 \\ 2 & 5 & 0 \\ 6 & 2 & -1 \end{vmatrix}$.

VI. (6) Let $A = \begin{bmatrix} t & 1 & 2 \\ -t & 1 & 1 \\ 0 & 2 & t \end{bmatrix}$.

(a) Calculate that $\det(A) = 2t(t-3)$ by using *cofactor expansion* of the determinant *down the first column.*

(b) Use the expression for $\det(A)$ in part (a) (even if you did not carry out the calculation) to determine the values of t for which A is singular.

VII. (4) As usual, let P_3 be the 4-dimensional vector space of polynomials of degree at most 3. Let $D\colon P_3 \to P_3$ be differentiation, a linear transformation. What is the kernel of D? What is the range of D? (Remark: there is no need to use a matrix representation. Just think about what polynomials are in the kernel and range.)

VIII. Let $L\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by
(15)

$$L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2x - z \\ 2x - y \\ x + 2y - z \end{bmatrix}$$

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(a) Find the *standard* matrix representation for L, that is, find the matrix A so that $AX = L(X)$ for every X in $\mathbb{R}^3$.

(b) Consider the ordered basis $S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\}$ for $\mathbb{R}^3$. By solving the system

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \lambda_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} + \lambda_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

for λ_1, λ_2, and λ_3, obtain a formula of the form $\begin{bmatrix} a \\ b \\ c \end{bmatrix}_S = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix}$ to convert a vector v in $\mathbb{R}^3$ into its corresponding vector v_S.

(c) Check that the formula that you found in part (b) really does convert the three vectors in the basis S to the standard basis vectors, that is, it should tell you that $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}_S = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and similarly for the second and third vectors in S. If it does not, go back and do part (b) correctly.

(d) Find the matrix representation of L *with respect to the basis* S given in part (b). That is, find the matrix A so that $Av_S = (L(v))_S$.