Mathematics 6823-001 Name (please print) ______________________

Final Examination

May 13, 2009

Instructions: Insofar as possible, give brief, clear answers. Use major theorems when possible. Assume that homology and cohomology are with $\mathbb{Z}$ coefficients unless otherwise indicated.

I. (6) Let C be a chain complex and let $[\varphi] \in H^n(C; G)$.

(a) Use the fact that φ is a cocycle to show that φ induces a homomorphism $\overline{\varphi|_{Z_n}} \colon H_n(C) \to G$.

(b) Show that if φ is a coboundary, then $\overline{\varphi}$ is the zero homomorphism. That is, sending the cohomology class $[\varphi]$ to $\overline{\varphi}$ is a well-defined homomorphism $h \colon H^n(C; G) \to \mathrm{Hom}(H_n(C), G)$.

II. (9) Let Y be the space obtained from the 3-sphere S^3 by attaching a 4-cell using a map of degree 6. It has a CW-complex structure with one cell in each of the dimensions 0, 3, and 4.

(a) Use cellular homology to calculate the homology of Y with $\mathbb{Z}$ coefficients.

(b) Use the Universal Coefficient Theorem to calculate the cohomology of Y with $\mathbb{Z}$ coefficients. You may use the fact that $\mathrm{Ext}(\mathbb{Z}/m, G) \cong G/mG$, so $\mathrm{Ext}(\mathbb{Z}/m, \mathbb{Z}) \cong \mathbb{Z}/m$.

(c) Use the Universal Coefficient Theorem to calculate the cohomology of Y with $\mathbb{Z}/3$ coefficients. You may use the fact that $\mathrm{Ext}(\mathbb{Z}/m, G) \cong G/mG$, so $\mathrm{Ext}(\mathbb{Z}/m, \mathbb{Z}/n) \cong \mathbb{Z}/\gcd(m, n)$.

III. (6) Give an example of a short exact sequence $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ of abelian groups that does not split. Give an example of a short exact sequence of nonabelian groups so that β has a section, but B is not the direct product of A and C.

IV. (8) Regarding the Klein bottle K as two Möbius bands glued together along their boundaries, use the Mayer-Vietoris sequence to calculate the homology groups of K.

V. (6) Give the definitions of $\varphi \cup \psi$, where $\varphi \in C^k(X; G)$ and $\psi \in C^\ell(X; G)$, and of $\sigma \cap \varphi$, where σ is a singular $(k+\ell)$-simplex in X.

VI. (6) Use Poincaré Duality to show that if M is a closed odd-dimensional manifold, then the Euler characteristic of M is 0. You may use the fact that $H^i(M; F) \cong \mathrm{Hom}(H_i(M; F), F)$ when F is a field, and also the fact that the $H_i(M; F)$ are finite-dimensional.

VII. (7) Recall that if X and Y are CW-complexes and each $H^k(Y; R)$ is free and finitely generated (as an R-module), then $H^*(X \times Y; R) \cong H^*(X; R) \otimes H^*(Y; R)$ as graded rings. Take as known the fact that for $n \geq 1$, $H^*(S^n) \cong H^0(S^n) \oplus H^n(S^n)$, with $H^0(S^n) \cong \mathbb{Z}$ generated by 1, $H^n(S^n) \cong \mathbb{Z}$ generated by an element α_n, and $\alpha_n \cup \alpha_n = 0$.

(a) Use this theorem to write down the cohomology ring $H^*(S^n \times S^n)$.

(b) Tell a ring isomorphism (not a graded ring isomorphism) from $H^*(S^2 \times S^2)$ to $H^*(S^4 \times S^4)$. You do not need to verify that it is is an isomorphism, just write it down.

(c) Show that $H^*(S^2 \times S^2)$ and $H^*(S^3 \times S^3)$ are not isomorphic as rings.

VIII. (6) Let U be an open subset of an R-orientable n-manifold M, and let $\{\mu_x\}_{x \in M}$ be an R-orientation of M. Verify that $\{\mu_x\}_{x \in U}$ is an R-orientation of U (i. e. is locally consistent for U).

IX. Let M be a *closed* connected R-orientable n-manifold. What is a *fundamental class* for M (with R
(6) coefficients)? State Poincaré Duality for this closed M in terms of a fundamental class.

X. Let $f\colon (I^n, \partial I^n) \to (X, x_0)$ represent an element of $\pi_n(X, x_0)$, and let $\omega\colon (I, \partial I) \to (X, x_0)$ be a loop based
(6) at x_0. Draw a picture and use it to describe a function $\omega f\colon (I^n, \partial I^n) \to (X, x_0)$ that represents the result $[\omega] \cdot [f]$ of the element $[\omega] \in \pi_1(X, x_0)$ acting on the element $[f] \in \pi_n(X, x_0)$. Do the same if one is thinking of f as a map from $(S^n, s_0) \to (X, x_0)$.

XI. Let M be a simply-connected n-manifold. Show that M is orientable. (Hint: what do we know about the
(6) covering spaces of a simply-connected space?)