Mathematics 6823-001 Name (please print) ____________________

Final Examination

May 13, 2009

Instructions: Insofar as possible, give brief, clear answers. Use major theorems when possible. Assume that homology and cohomology are with $\mathbb{Z}$ coefficients unless otherwise indicated.

I. (6) Let C be a chain complex and let $[\varphi] \in H^n(C;G)$.

(a) Use the fact that φ is a cocycle to show that φ induces a homomorphism $\overline{\varphi|_{Z_n}} \colon H_n(C) \to G$.

The cocycle condition $0 = \delta_n\varphi = \varphi\partial_{n+1}$ says that $0 = \varphi(\partial_{n+1}(C_{n+1})) = \varphi(B_n)$, so $\varphi|_{Z_n} \colon Z_n \to G$ induces $\overline{\varphi|_{Z_n}} \colon Z_n/B_n = H_n(C) \to G$.

(b) Show that if φ is a coboundary, then $\overline{\varphi}$ is the zero homomorphism. That is, sending the cohomology class $[\varphi]$ to $\overline{\varphi}$ is a well-defined homomorphism $h \colon H^n(C;G) \to \operatorname{Hom}(H_n(C), G)$.

If $\varphi = \delta\phi = \phi \circ \partial_n$, then $\varphi|_{Z_n}(Z_n) = \phi \circ \partial_n(Z_n) = \phi(0) = 0$.

II. (9) Let Y be the space obtained from the 3-sphere S^3 by attaching a 4-cell using a map of degree 6. It has a CW-complex structure with one cell in each of the dimensions 0, 3, and 4.

(a) Use cellular homology to calculate the homology of Y with $\mathbb{Z}$ coefficients.

The cellular chain complex of Y terminates with

$$0 \to \mathbb{Z} \xrightarrow{\partial_4} \mathbb{Z} \to 0 \to 0 \to \mathbb{Z} \to 0$$

Clearly we have $H_0(Y) \cong \mathbb{Z}$. Since ∂_4 is multiplication by 6, its kernel $H_4(Y)$ is 0 and its cokernel $H_3(Y)$ is $\mathbb{Z}/6$.

(b) Use the Universal Coefficient Theorem to calculate the cohomology of Y with $\mathbb{Z}$ coefficients. You may use the fact that $\operatorname{Ext}(\mathbb{Z}/m, G) \cong G/mG$, so $\operatorname{Ext}(\mathbb{Z}/m, \mathbb{Z}) \cong \mathbb{Z}/m$.

From

$$0 \to \operatorname{Ext}(H_{-1}(Y), \mathbb{Z}) \to H^0(Y) \to \operatorname{Hom}(H_0(Y), \mathbb{Z}) \to 0$$

we have $H^0(Y) \cong \operatorname{Hom}(H_0(Y), \mathbb{Z}) \cong \operatorname{Hom}(\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}$. From

$$0 \to \operatorname{Ext}(H_2(Y), \mathbb{Z}) \to H^3(Y) \to \operatorname{Hom}(H_3(Y), \mathbb{Z}) \to 0$$

we have $H^3(Y) \cong \operatorname{Hom}(H_3(Y), \mathbb{Z}) \cong \operatorname{Hom}(\mathbb{Z}/6, \mathbb{Z}) \cong 0$. From

$$0 \to \operatorname{Ext}(H_3(Y), \mathbb{Z}) \to H^4(Y) \to \operatorname{Hom}(H_4(Y), \mathbb{Z}) \to 0$$

we have $H^4(Y) \cong \operatorname{Ext}(H_3(Y), \mathbb{Z}) \cong \operatorname{Ext}(\mathbb{Z}/6, \mathbb{Z}) \cong \mathbb{Z}/6$.

(c) Use the Universal Coefficient Theorem to calculate the cohomology of Y with $\mathbb{Z}/3$ coefficients. You may use the fact that $\operatorname{Ext}(\mathbb{Z}/m, G) \cong G/mG$, so $\operatorname{Ext}(\mathbb{Z}/m, \mathbb{Z}/n) \cong \mathbb{Z}/\gcd(m,n)$.

From

$$0 \to \operatorname{Ext}(H_{-1}(Y), \mathbb{Z}/3) \to H^0(Y;\mathbb{Z}/3) \to \operatorname{Hom}(H_0(Y), \mathbb{Z}/3) \to 0$$

we have $H^0(Y;\mathbb{Z}/3) \cong \operatorname{Hom}(H_0(Y), \mathbb{Z}/3) \cong \operatorname{Hom}(\mathbb{Z}, \mathbb{Z}/3) \cong \mathbb{Z}/3$. From

$$0 \to \operatorname{Ext}(H_2(Y), \mathbb{Z}/3) \to H^3(Y;\mathbb{Z}/3) \to \operatorname{Hom}(H_3(Y), \mathbb{Z}/3) \to 0$$

we have $H^3(Y;\mathbb{Z}/3) \cong \operatorname{Hom}(H_3(Y), \mathbb{Z}/3) \cong \operatorname{Hom}(\mathbb{Z}/6, \mathbb{Z}/3) \cong \mathbb{Z}/3$. From

$$0 \to \operatorname{Ext}(H_3(Y), \mathbb{Z}/3) \to H^4(Y;\mathbb{Z}/3) \to \operatorname{Hom}(H_4(Y), \mathbb{Z}/3) \to 0$$

we have $H^4(Y;\mathbb{Z}/3) \cong \operatorname{Ext}(H_3(Y), \mathbb{Z}/3) \cong \operatorname{Ext}(\mathbb{Z}/6, \mathbb{Z}/3) \cong \mathbb{Z}/3$.

III. (6) Give an example of a short exact sequence $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ of abelian groups that does not split. Give an example of a short exact sequence of nonabelian groups so that β has a section, but B is not the direct product of A and C.

The short exact sequence $0 \to \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \xrightarrow{\beta} \mathbb{Z}/2 \to 0$, where α is multiplication by 2, does not split. The reason is that $\mathbb{Z}$ is torsion-free and therefore the only homomorphism s from $\mathbb{Z}/2$ to $\mathbb{Z}$ is the zero homomorphism. So one cannot have βs equal to the identity on $\mathbb{Z}/2$.

The short exact sequence $0 \to C_3 \xrightarrow{\alpha} \Sigma_3 \xrightarrow{\beta} C_2 \to 0$ (where Σ_3 is the permutation group on three letters) has a section $s\colon C_2 \to \Sigma_3$ given by sending the generator to any transposition. But Σ_3 is not isomorphic to the direct sum $C_2 \oplus C_3$, since the latter is abelian.

IV. (8) Regarding the Klein bottle K as two Möbius bands glued together along their boundaries, use the Mayer-Vietoris sequence to calculate the homology groups of K.

Let A and B be the two Möbius bands, which intersect in their common boundary circle C. Since K is connected, $H_0(K) \cong \mathbb{Z}$. For its other homology groups, we use the reduced Mayer-Vietoris sequence

$$0 \to H_2(K) \to H_1(C) \xrightarrow{\Phi} H_1(A) \oplus H_1(B) \to H_1(K) \to 0$$

Since A and B are homotopy equivalent to circles, this becomes

$$0 \to H_2(K) \to \mathbb{Z} \xrightarrow{\Phi} \mathbb{Z} \oplus \mathbb{Z} \to H_1(K) \to 0$$

The generator of $H_1(C)$ goes to twice a generator of $H_1(A)$ under inclusion, and similarly for $H_1(B)$, so we can write $\Phi(1) = (2, -2)$. Therefore Φ is injective, showing that $H_2(K) = 0$, and $H_1(K) \cong \mathbb{Z} \oplus \mathbb{Z}/\langle(2, -2)\rangle \cong \mathbb{Z} \oplus \mathbb{Z}/2$, where the generators of the quotient are $\overline{(1, 0)}$ and $\overline{(1, -1)}$.

V. (6) Give the definitions of $\varphi \cup \psi$, where $\varphi \in C^k(X; G)$ and $\psi \in C^\ell(X; G)$, and of $\sigma \cap \varphi$, where σ is a singular $(k + \ell)$-simplex in X.

Writing $\sigma\colon \Delta^{k+\ell} \to X$, where $\Delta^{k+\ell} = [v_0, \ldots, v_{k+\ell}]$, we have

$$\varphi \cup \psi(\sigma) = \varphi(\sigma|_{[v_0, \ldots, v_k]}) \cdot \psi(\sigma|_{[v_k, \ldots, v_{k+\ell}]})$$

and

$$\sigma \cap \varphi = \varphi(\sigma|_{[v_0, \ldots, v_k]}) \cdot \sigma|_{[v_k, \ldots, v_{k+\ell}]}$$

VI. (6) Use Poincaré Duality to show that if M is a closed odd-dimensional manifold, then the Euler characteristic of M is 0. You may use the fact that $H^i(M;F) \cong \mathrm{Hom}(H_i(M;F),F)$ when F is a field, and also the fact that the $H_i(M;F)$ are finite-dimensional.

We take $R = \mathbb{Z}/2$, so that M is $\mathbb{Z}/2$-orientable, each $H_i(M)$ is a finite-dimensional $\mathbb{Z}/2$-vector space, and $H^k(M;\mathbb{Z}/2) \cong \mathrm{Hom}(H_k(M;\mathbb{Z}/2),\mathbb{Z}/2)$, so $\dim(H^k(M;\mathbb{Z}/2)) = \dim(H_k(M;\mathbb{Z}/2))$. By Poincaré Duality, $H^k(M;\mathbb{Z}/2) \cong H_{n-k}(M;\mathbb{Z}/2)$ and hence $\dim(H_k(M;\mathbb{Z}/2)) = \dim(H^k(M;\mathbb{Z}/2)) = \dim(H_{n-k}(M;\mathbb{Z}/2))$. So we have

$$\chi(M) = \sum_{k=0}^{n} (-1)^i \dim(H_i(M\ \mathbb{Z}/2)) = \sum_{k=0}^{(n-1)/2} (-1)^i \dim(H_i(M;\mathbb{Z}/2)) + (-1)^{n-i} \dim(H_{n-i}(M;\mathbb{Z}/2))$$
$$= \sum_{k=0}^{(n-1)/2} ((-1)^i + (-1)^{n-i}) \dim(H_i(M;\mathbb{Z}/2)) = \sum_{k=0}^{(n-1)/2} (-1)^i(1+(-1)^n) \dim(H_i(M;\mathbb{Z}/2)) = 0\ .$$

VII. (7) Recall that if X and Y are CW-complexes and each $H^k(Y;R)$ is free and finitely generated (as an R-module), then $H^*(X \times Y;R) \cong H^*(X;R) \otimes H^*(Y;R)$ as graded rings. Take as known the fact that for $n \geq 1$, $H^*(S^n) \cong H^0(S^n) \oplus H^n(S^n)$, with $H^0(S^n) \cong \mathbb{Z}$ generated by 1, $H^n(S^n) \cong \mathbb{Z}$ generated by an element α_n, and $\alpha_n \cup \alpha_n = 0$.

(a) Use this theorem to write down the cohomology ring $H^*(S^n \times S^n)$.

Denote the n-dimensional generators of the two factors by α_n and β_n. The nonzero cohomology groups are
$H^0(S^n \times S^n) \cong \mathbb{Z}$ generated by $1 \cup 1 = 1$,
$H^n(S^n \times S^n) \cong \mathbb{Z} \oplus \mathbb{Z}$ generated by $\alpha_n \cup 1 = \alpha_n$ and $1 \cup \beta_n = \beta_n$,
and $H^{2n}(S^n \times S^n) \cong \mathbb{Z}$ generated by $\alpha_n \cup \beta_n$.

(b) Tell a ring isomorphism (not a graded ring isomorphism) from $H^*(S^2 \times S^2)$ to $H^*(S^4 \times S^4)$. You do not need to verify that it is is an isomorphism, just write it down.

Send α_2 to α_4 and β_2 to β_4.

(c) Show that $H^*(S^2 \times S^2)$ and $H^*(S^3 \times S^3)$ are not isomorphic as rings.

All nonzero elements of $H^*(S^2 \times S^2)$ occur in even dimensions, so any two commute. But in $H^*(S^3 \times S^3)$, $\alpha_3 \cup \beta_3 = -\beta_3 \cup \alpha_3$.

VIII. (6) Let U be an open subset of an R-orientable n-manifold M, and let $\{\mu_x\}_{x\in M}$ be an R-orientation of M. Verify that $\{\mu_x\}_{x\in U}$ is an R-orientation of U (i. e. is locally consistent for U).

Let $x \in U$. Since $\{\mu_x\}_{x\in M}$ is locally consistent, there is an open n-ball B in M, containing x, and an element $\mu_B \in H_n(M, M-B;R)$ such that $H_n(M, M-B;R) \to H_n(M, M-y;R)$ carries μ_B to μ_y for every $y \in B$. Since U is open, there is an open n-ball B' in U with $x \in B'$ and $\overline{B'} \subset B$. Since $M - B'$ deformation retracts to $M - B$, inclusion induces an isomorphism $H_n(M, M-B;R) \cong H_n(M, M-B';R)$, which sends μ_B to some element $\mu_{B'}$. For each $y \in B$, the inclusions induce isomorphisms $H_n(M, M-B;R) \cong H_n(M, M-B';R) \cong H_n(M, M-y;R)$, and μ_B goes to μ_y, showing that $\mu_{B'}$ goes to μ_y. Therefore $\{\mu_x\}_{x\in U}$ is also locally consistent.

IX. (6) Let M be a *closed* connected R-orientable n-manifold. What is a *fundamental class* for M (with R coefficients)? State Poincaré Duality for this closed M in terms of a fundamental class.

A fundamental class $[M]$ is a generator of $H_n(M;R) \cong R$. Poincaré Duality says that the R-homomorphism $D_M\colon H^k(M;R) \to H_{n-k}(M;R)$ given by $D_M(\alpha) = [M] \cap \alpha$ is an isomorphism.

X. (6) Let $f\colon (I^n, \partial I^n) \to (X, x_0)$ represent an element of $\pi_n(X, x_0)$, and let $\omega\colon (I, \partial I) \to (X, x_0)$ be a loop based at x_0. Draw a picture and use it to describe a function $\omega f\colon (I^n, \partial I^n) \to (X, x_0)$ that represents the result $[\omega] \cdot [f]$ of the element $[\omega] \in \pi_1(X, x_0)$ acting on the element $[f] \in \pi_n(X, x_0)$. Do the same if one is thinking of f as a map from $(S^n, s_0) \to (X, x_0)$.

(For the I^n case, draw the standard picture where f is a smaller square inside a larger one, and each radial line between the boundaries of the two squares is sent to X by mapping it around the loop ω. For the sphere viewpoint, draw the "balloon" picture and a partial quotient map from S^n to the balloon, with the balloon part then mapping by f and the "string" mapping around ω.)

XI. (6) Let M be a simply-connected n-manifold. Show that M is orientable. (Hint: what do we know about the covering spaces of a simply-connected space?)

M cannot have a connected 2-fold covering, since then $\pi_1(M)$ would have to contain a subgroup of index 2. Therefore the 2-fold orientation covering of M has two components, so M is orientable.