Mathematics 6823-001 Name (please print) ______________________

Midterm Examination

March 12, 2009

Instructions: Insofar as possible, give brief, clear answers. Use major theorems when possible.

I. (6) Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be an exact sequence of abelian groups, and let G be an abelian group. Give an example showing that the sequence $0 \to \mathrm{Hom}(C,G) \xrightarrow{g^*} \mathrm{Hom}(B,G) \xrightarrow{f^*} \mathrm{Hom}(A,G) \to 0$ need not be exact. What positive statement can be made?

Choose $m > 1$ and consider $0 \to \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \to \mathbb{Z}/m \to 0$, where $\alpha(k) = mk$. Now $\mathrm{Hom}(\mathbb{Z},\mathbb{Z}) \cong \mathbb{Z}$ generated by $\phi\colon 1 \mapsto 1$, and $\alpha^*(\phi)\colon 1 \mapsto m$ so $\alpha^*(\phi) = m\phi$. Therefore the sequence $0 \to \mathrm{Hom}(\mathbb{Z}/m,\mathbb{Z}) \to \mathrm{Hom}(\mathbb{Z},\mathbb{Z}) \xrightarrow{\alpha^*} \mathrm{Hom}(\mathbb{Z},\mathbb{Z}) \to 0$ becomes $0 \to 0 \to \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \to 0$, which is not exact.

It is true, however, that $0 \to \mathrm{Hom}(C,G) \xrightarrow{g^*} \mathrm{Hom}(B,G) \xrightarrow{f^*} \mathrm{Hom}(A,G)$ is exact. And the Hom sequence is exact if the original exact sequence was split exact.

II. (6) Let X be obtained from the 2-sphere by identifying three points of the equator. Compute the homology groups of X. (Note that X has a cell structure with one 0-cell, three 1-cells, and two 2-cells.)

Form the cellular chain complex

$$0 \to \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\partial_2} \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\partial_1} \mathbb{Z} \to 0\ .$$

with the obvious bases consisting of the cells. The matrix of ∂_2 is $\begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}$, which is equivalent to $\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$, so $H_2(X) = \ker(\partial_2) \cong \mathbb{Z}$. Since there is only one 0-cell, ∂_1 is the zero homomorphism, so $H_1(X) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} / \mathrm{im}(\partial_2) \cong \mathbb{Z} \oplus \mathbb{Z}$, and $H_0(X) = \mathbb{Z} / \mathrm{im}(\partial_1) = \mathbb{Z}$.

Alternatively, one can put A equal to the subset consisting of the three points and use the fact that $H_n(S^2, A) \cong \widetilde{H}_n(S^2/A)$. The long exact sequence becomes

$$H_2(A) = 0 \to H_2(S^2) \to H_2(S^2/A) \to H_1(A) \to H_1(S^2) \to H_1(S^2/A) \to \widetilde{H}_0(A) \to 0 = \widetilde{H}_0(S^2)$$

Since $H_1(A) = 0$, we have $H_2(S^2/A) \cong H_2(S^2) \cong \mathbb{Z}$. Since $H_1(S^2) = 0$, we have $H_1(S^2/A) \cong \widetilde{H}_0(A) \cong \mathbb{Z} \oplus \mathbb{Z}$, and $H_0(S^2/A) \cong \mathbb{Z}$ since S^2 and hence S^2/A is path connected.

III. (6) Let X be a finite CW-complex, and let A and B be subcomplexes of X with $X = A \cup B$. Explain why the Euler characteristic satisfies $\chi(X) = \chi(A) + \chi(B) - \chi(A \cap B)$.

$\chi(X)$ is the alternating sum $\sum(-1)^i x_i$, where x_i is the number of i-cells of X. Similarly, $\chi(A) = \sum(-1)^i a_i$ and $\chi(B) = \sum(-1)^i b_i$. In the sum $\chi(A) + \chi(B)$, all the cells of X are counted with the correct sign for the expression for $\chi(X)$, but the cells in $A \cap B$ are counted twice. Correcting this by subtracting $\chi(A \cap B)$ gives the formula.

IV. (6) Let C be a chain complex and let $[\varphi] \in H^n(C;G)$.

(a) Use the fact that φ is a cocycle to show that φ induces a homomorphism $\overline{\varphi|_{Z_n}}\colon H_n(C) \to G$.

The cocycle condition $0 = \delta_n\varphi = \varphi\partial_{n+1}$ says that $0 = \varphi(\partial_{n+1}(C_{n+1})) = \varphi(B_n)$, so $\varphi|_{Z_n}\colon Z_n \to G$ induces $\overline{\varphi|_{Z_n}}\colon Z_n/B_n = H_n(C) \to G$.

(b) Show that if φ is a coboundary, then $\overline{\varphi}$ is the zero homomorphism. That is, sending the cohomology class $[\varphi]$ to $\overline{\varphi}$ is a well-defined homomorphism $h\colon H^n(C;G) \to \mathrm{Hom}(H_n(C),G)$.

If $\varphi = \delta\phi = \phi \circ \partial_n$, then $\varphi|_{Z_n}(Z_n) = \phi \circ \partial_n(Z_n) = \phi(0) = 0$.

V. (6) Let H be an abelian group (or more generally an R-module over a ring R). Define a *free resolution* of H. Suppose that F and F' are free resolutions of H and H', and $\alpha\colon H \to H'$ is a homomorphism. Tell what is obtained from α, and how well-defined it is.

A free resolution of H is an exact sequence $\cdots \to F_3 \xrightarrow{f_3} F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_0} H \to 0$, where each F_i is a free R-module.

From α one can always build a chain map

$$\begin{array}{ccccccccccc}
\cdots \longrightarrow & F_2 & \xrightarrow{f_2} & F_1 & \xrightarrow{f_1} & F_0 & \xrightarrow{f_0} & H & \longrightarrow & 0 \\
 & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_0} & & \downarrow{\scriptstyle \alpha} & & \\
\cdots \longrightarrow & F_2' & \xrightarrow{f_2'} & F_1' & \xrightarrow{f_1'} & F_0' & \xrightarrow{f_0'} & H' & \longrightarrow & 0
\end{array}$$

which is well-defined up to chain homotopy.

VI. (8) State the Excision Theorem (either of the two forms is sufficient). Use it to calculate $H_n(U, U - x)$, where U is an open subset of $\mathbb{R}^n$ and $x \in U$.

For the statements, check the text. Given $x \in U$, choose a small closed n-ball B in U with x in the interior of B. Since $\overline{U - B} \subset U - x$, we have by excision that the inclusion $(B, B - x) = (U - U - B, U - x - (U - B)) \to (U, U - x)$ induces an isomorphism on homology groups. So for each k, we have $H_k(U, U - x) \cong H_k(B, B - x) \cong H_k(B, \partial B)$, since $B - x$ deformation retracts to ∂B. Since all reduced homology groups of B are 0, the long exact sequence gives an isomorphism $H_k(B, \partial B) \cong \widetilde{H}_{k-1}(\partial B) \cong \widetilde{H}_{k-1}(S^{n-1})$, which is $\mathbb{Z}$ if $k = n$ and 0 otherwise.

I use the previous argument because it actually works in any manifold. But a nicer way to do this particular case is to take $Z = \mathbb{R}^n - U$ and use the excision $(U, U - x) \to (\mathbb{R}^n, \mathbb{R}^n - x)$. Then one has $H_k(U, U - x) \cong H_k(\mathbb{R}^n, \mathbb{R}^n - x) \cong \widetilde{H}_{k-1}(\mathbb{R}^n - x) \cong \widetilde{H}_{k-1}(S^{n-1})$ which is $\mathbb{Z}$ if $k = n$ and 0 otherwise.

VII. (4) Construct a surjective map of degree 0 from S^n to S^n.

Regarding S^n as the standard subset in $\mathbb{R}^{n+1}$, the projection p of S^n to the first n coordinates carries S^n onto $D^n \subset \mathbb{R}^n$. Following this by the quotient map $q\colon D^n \to D^n/\partial D^n = S^n$ defines the surjection $qp\colon S^n \to S^n$. It has degree 0 since $(qp)_*$ factors as $H_n(S^n) \xrightarrow{p_*} H_n(D^n) \xrightarrow{q_*} H_n(S^n)$, and $H_n(D^n) = 0$.

VIII. (8) Define the terms *category, covariant functor, and contravariant functor.* Give an elementary (undergraduate) example of a contravariant functor.

A *category* C consists of a set of objects $\mathrm{Obj}(C)$ and, for each $X, Y \in C$, a set of morphisms $\mathrm{Mor}(X, Y)$, such that $\mathrm{Mor}(X, X)$ contains an identity morphisms ι_X. Moreover, for any $X, Y, Z \in C$, there is a composition function $\circ\colon \mathrm{Mor}(Y, Z) \times \mathrm{Mor}(X, Y) \to \mathrm{Mor}(X, Z)$ such that for $f \in \mathrm{Mor}(X, Y)$ and $g \in \mathrm{Mor}(Y, Z)$, $(f \circ g) \circ h = f \circ (g \circ h)$, $f \circ \iota_X = f$, and $\iota_Y \circ f = f$.

A covariant functor $F\colon C \to D$ assigns to each object $X \in \mathrm{Obj}(C)$ an object $F(X) \in \mathrm{Obj}(D)$ and to each $f \in \mathrm{Mor}(X, Y)$ a morphism $F(f) \in \mathrm{Mor}(F(X), F(Y))$ such that $F(\iota_X) = \iota_{F(X)}$ and $F(g \circ f) = F(g) \circ F(f)$. A contravariant functor is the same, except that $F(g \circ f) = F(f) \circ F(g)$.

An elementary example of a contravariant functor is taking both categories to be the category of real vector spaces and linear transformations, and putting $F(V)$ equal to the dual space $\mathrm{Hom}(V, \mathbb{R})$ and, for $f \in \mathrm{Hom}(V, W)$, $F(f)$ equal to the dual linear transformation $f^*\colon \mathrm{Hom}(W, \mathbb{R}) \to \mathrm{Hom}(V, \mathbb{R})$ given by $f^*(\phi) = \phi \circ f$.