Mathematics 3333-002 Name (please print) ____________________

Final Examination Form A

May 11, 2010

Instructions: Give concise answers, but clearly indicate your reasoning.

I. (4) Let V be an inner product space, that is, a vector space V equipped with an inner product denoted by (u, v). Let v_0 be a fixed vector, and let $W = \{v \in V \mid (v, v_0) = 0\}$ be the set of vectors in V that are orthogonal to v_0. Verify that W is a subspace of V.

Suppose that w_1 and w_2 are in W. That is, $(w_1, v_0) = 0$ and $(w_2, v_0) = 0$. Then we have $(w_1 + w_2, v_0) = (w_1, v_0) + (w_2, v_0) = 0 + 0 = 0$, so $w_1 + w_2$ is in W. Also, if λ is any scalar, then $(\lambda w_1, v_0) = \lambda(w_1, v_0) = \lambda \cdot 0 = 0$ so λw_1 is in W.

II. (4) Let θ be a fixed real number, and in $\mathbb{R}^2$ let $v_\theta = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}$ and $w_\theta = \begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \end{bmatrix}$. Assuming that $\mathbb{R}^2$ has the standard inner product, verify that $\{v_\theta, w_\theta\}$ is an orthonormal basis for $\mathbb{R}^2$.

v_θ and w_θ are nonzero and $(v_\theta, w_\theta) = \cos(\theta)(-\sin(\theta)) + \sin(\theta)\cos(\theta) = 0$, so $\{v_\theta, w_\theta\}$ is an orthogonal set of vectors. Therefore it is linearly independent. Since the dimension of $\mathbb{R}^2$ is 2, $\{v_\theta, w_\theta\}$ is a basis of $\mathbb{R}^2$.

We also have $\| v_\theta \|^2 = (v_\theta, v_\theta) = \cos^2(\theta) + \sin^2(\theta) = 1$, so $\| v_\theta \|^2 = 1$, and $\| w_\theta \|^2 = (w_\theta, w_\theta) = (-\sin(\theta))^2 + \cos^2(\theta) = \sin^2(\theta)^2 + \cos^2(\theta) = 1$, so $\| w_\theta \|^2 = 1$. So $\{v_\theta, w_\theta\}$ is an orthonormal basis of $\mathbb{R}^2$.

III. (6) Let $f\colon V \to W$ be a linear transformation between two vector spaces.

(a) Define the *kernel* of f.

The kernel of f is $\{v \text{ in } V \mid f(v) = 0\}$.

(b) Verify that the kernel of f is a subspace of V.

Suppose that v_1 and v_2 are in the kernel. Then $f(v_1 + v_2) = f(v_1) + f(v_2) = 0$, so $v_1 + v_2$ is in the kernel. For any scalar λ, $f(\lambda v_1) = \lambda f(v_1) = \lambda \cdot 0 = 0$, so λv_1 is in the kernel.

IV. (6) Each of the following matrices is the augmented matrix of a system of linear equations, and is in row echelon form or reduced row echelon form. For each matrix, use back substitution to write a general expression for the solutions of the corresponding linear system.

1. $\begin{bmatrix} 0 & 1 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

x_1, x_3, x_4, and x_5 are free parameters, and the first equation says that $x_2 = x_3 + x_4 + x_5 - 1$, so the general solution is $(r, s + t + u - 1, s, t, u)$.

2. $\begin{bmatrix} 1 & a & b & 1 \\ 0 & 1 & c & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ (the answer will involve a, b, and c)

x_3 is a free parameter, the second equation says that $x_2 = 1 - cx_3$, and the first says that $x_1 = 1 - ax_2 - bx_3 = 1 - a(1 - cx_3) - bx_3 = 1 - a + acx_3 - bx_3 = 1 - a + (ac - b)x_3$, so the general solution is $(1 - a + (ac - b)r, 1 - cr, r)$.

V. (6) Let $A = \begin{bmatrix} 0 & 0 & -9 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.

(a) Calculate the characteristic polynomial of A.

$$\det(\lambda I_3 - A) = \begin{vmatrix} \lambda & 0 & 9 \\ 0 & \lambda - 1 & 0 \\ -1 & 0 & \lambda \end{vmatrix} = (\lambda - 1) \begin{vmatrix} \lambda & 9 \\ -1 & \lambda \end{vmatrix} = (\lambda - 1)(\lambda^2 + 9)$$

(b) Use the characteristic polynomial to find the only (real) eigenvalue of A.

The only (real) root is $\lambda = 1$, so this is the only eigenvalue.

(c) Find an eigenvector for the eigenvalue.

We examine the null space of $\lambda I_3 - A$ when $\lambda = 1$:

$$I_3 - A = \begin{bmatrix} 1 & 0 & 9 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 9 \\ 0 & 0 & 10 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 9 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

so the null space is all vectors of the form $\begin{bmatrix} 0 \\ r \\ 0 \end{bmatrix}$, and a 1-eigenvector is $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

VI. (4) Let $A = ([a_{i,j}])$ be a 5×5 matrix, and consider the formula

$$\det(A) = \sum (\pm) a_{1,\sigma(1)} a_{2,\sigma(2)} a_{3,\sigma(3)} a_{4,\sigma(4)} a_{5,\sigma(5)} \ .$$

Determine the sign (i. e. tell whether the term has a plus or a minus sign in the formula) of the term that contains $a_{1,2}a_{2,4}a_{3,5}a_{4,3}a_{5,1}$ (make your reasoning clear— answers of "plus" or "minus" without a correct explanation won't receive any credit).

The permutation 24531 has $1+2+2+1 = 6$ inversions, so is even, so the term that contains $a_{1,2}a_{2,4}a_{3,5}a_{4,3}a_{5,1}$ has a plus sign.

VII. (15) Label each of the following statements either T for true or F for false. The symbol V that appears in some of the statements denotes a finite-dimensional vector space, and $\{v_1, \ldots, v_k\}$ denotes a finite subset of V.

__T__ If a 5×5 matrix has 5 distinct eigenvalues, then it must be diagonalizable.

__T__ If $\{v_1, \ldots, v_k\}$ spans V, then some subset of $\{v_1, \ldots, v_k\}$ is a basis for V.

__T__ If $\{v_1, \ldots, v_k\}$ is a basis for V, and two linear transformation f and g from V to W satisfy $f(v_i) = g(v_i)$ for $1 \le i \le k$, then $f(v) = g(v)$ for every v in V.

__T__ When a homogeneous linear system is written in matrix form $AX = 0$, the null space of A equals the solution space of the linear system.

__T__ If $(\_, \_)$ is an inner product on $\mathbb{R}^n$, and $c_{ij} = (e_i, e_j)$, then the matrix $C = [c_{ij}]$ satisfies $(v, w) = v^T C w$ for any two vectors v and w in $\mathbb{R}^n$.

__F__ The range of a matrix transformation equals the row space of the matrix.

__F__ When a homogeneous linear system is written in matrix form $AX = 0$, the rank of A equals the dimension of the solution space of the linear system.

__T__ Similar matrices must have the same characteristic polynomial.

__T__ The only matrix that is similar to the identity matrix is the identity matrix.

__T__ If a matrix is in row echelon form, then its nonzero rows are linearly independent.

__F__ If a matrix is in row echelon form, then its nonzero columns are linearly independent.

__T__ A linear transformation is diagonalizable exactly when there is a basis consisting entirely of eigenvectors.

__T__ Coordinate vectors satisfy the formulas $(v + w)_S = v_S + w_S$ and $(\lambda v)_S = \lambda v_S$.

__T__ When a matrix $[a_{i,j}]$ is lower triangular, its characteristic polynomial is $(\lambda - a_{1,1})(\lambda - a_{2,2}) \cdots (\lambda - a_{n,n})$.

__F__ A matrix that has no eigenvectors must be singular.

VIII. (10) Let V be the 2-dimensional vector space consisting of solutions to the differential equation $y'' = y$. Recall that e^x, e^{-x}, $\cosh(x) = (e^x + e^{-x})/2$ and $\sinh(x) = (e^x - e^{-x})/2$ are well-known solutions of this equation. Let $S = \{e^x, e^{-x}\}$ and $T = \{\cosh(x), \sinh(x)\}$. These are bases of V (you do not need to verify this).

(a) Find $(2e^x - 3e^{-x})_S$ and $(2e^x - 3e^{-x})_T$.

$$(2e^x - 3e^{-x})_S = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

$2e^x - 3e^{-x} = a(e^x + e^{-x})/2 + b(e^x - e^{-x})/2 = (a/2 + b/2)e^x + (a/2 - b/2)e^{-x}$ gives $a + b = 4$, $a - b = -6$ so $a = -1$ and $b = 5$. Therefore $(2e^x - 3e^{-x})_T = \begin{bmatrix} -1 \\ 5 \end{bmatrix}$.

(b) Find the transition matrix $P_{S \leftarrow T}$, and verify that $P_{S \leftarrow T}(2e^x - 3e^{-x})_T = (2e^x - 3e^{-x})_S$.

Since $\cosh(x)_S = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$ and $\sinh(x)_S = \begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix}$, we have $P = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$.

$$\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} -1 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}.$$

(c) Let $D\colon V \to V$ be differentiation, $D(y) = y'$, which is a linear transformation. Find the representation matrix of D with respect to the basis S, and with respect to the basis T.

For the S-basis, $D(e^x) = e^x$ and $D(e^{-x}) = -e^{-x}$, so the representation matrix is $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

For the T-basis, $D(\cosh(x)) = \sinh(x)$ and $D(\sinh(x)) = \cosh(x)$, so the representation matrix is $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

IX. (9) A certain 3×3 matrix A has eigenvalues 2, 3, and -1.

A 2-eigenvector of A is $\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$, a 3-eigenvector of A is $\begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$, and a (-1)-eigenvector of A is $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

(a) Calculate $A\left(\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} - \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}\right)$.

$$A\left(\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} - \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}\right) = A\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} - A\begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} - 3\begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -6 \\ 9 \end{bmatrix}.$$

(b) Calculate $A\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

We express $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ in terms of the eigenvectors:

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 2 & 0 \\ 3 & -1 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & -2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

and then

$$A\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = A\left(\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}\right) = A\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + A\begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} - A\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \\ 4 \end{bmatrix}.$$

(c) Tell a 3×3 matrix P such that $P^{-1}AP$ is a diagonal matrix, and tell the diagonal matrix.

$$P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \\ 3 & -1 & 1 \end{bmatrix}, \text{ and } D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

X. (4) Recall that if $\{v_1, \ldots, v_k\}$ is a subset of a vector space V, $\operatorname{span}(\{v_1, \ldots, v_k\})$ is the set of linear combinations $\{\lambda_1 v_1 + \cdots + \lambda_k v_k \mid \text{ the } \lambda_i \text{ are numbers.}\}$. Verify that $\operatorname{span}(\{v_1, \ldots, v_k\})$ is a subspace of V.

Take any $\lambda_1 v_1 + \cdots + \lambda_k v_k$ and $\mu_1 v_1 + \cdots + \mu_k v_k$ in $\operatorname{span}(\{v_1, \ldots, v_k\})$. Adding these gives $(\lambda_1 + \mu_1)v_1 + \cdots + (\lambda_k + \mu_k)v_k$, which is also in $\operatorname{span}(\{v_1, \ldots, v_k\})$. Also, for any number λ, $\lambda(\lambda_1 v_1 + \cdots + \lambda_k v_k) = (\lambda\lambda_1)v_1 + \cdots + (\lambda\lambda_k)v_k$, which is again in $\operatorname{span}(\{v_1, \ldots, v_k\})$.

XI. (6) (a) Tell two elementary 3×3 matrices E and F such that $EFA = B$, where $A = \begin{bmatrix} 1 & -1 & 3 \\ 4 & 1 & -5 \\ 7 & 0 & 2 \end{bmatrix}$ and

$$B = \begin{bmatrix} 2 & -2 & 6 \\ 4 & 1 & -5 \\ 4 & 3 & -7 \end{bmatrix}$$

A is transformed to B by $R_3 - 3R_1 \to R_3$ followed by $2R_1 \to R_1$, so one choice is $E = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and

$F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$. Another choice is $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3/2 & 0 & 1 \end{bmatrix}$ and $F = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

(b) Is the matrix $\begin{bmatrix} 2 & -1 & 5 \\ 2 & 1 & -3 \\ 2 & 0 & 1 \end{bmatrix}$ a product of elementary matrices? Why or why not?

No, it is not. It is singular, as can be seen by calculating its determinant or by transforming it using elementary row operations to a matrix such as $\begin{bmatrix} 2 & -1 & 5 \\ 0 & 0 & 0 \\ 0 & 1 & -4 \end{bmatrix}$, and since elementary matrices are nonsingular, a product of elementary matrices would also be nonsingular.

XII. (a) Let $\{v_1, \ldots, v_k\}$ be a set of vectors in a vector space V. Define what it means to say that $\{v_1, \ldots, v_k\}$ is linearly independent.
(6)

It means that a linear combination $\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_k v_k$ can equal 0 only when every $\lambda_i = 0$.

(b) Test $\left\{ \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 8 \end{bmatrix} \right\}$ for linear independence.

Suppose that $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \lambda_1 \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} + \lambda_3 \begin{bmatrix} 4 \\ 0 \\ 8 \end{bmatrix} = \begin{bmatrix} 3\lambda_1 + 3\lambda_2 + 4\lambda_3 \\ 3\lambda_1 + \lambda_2 \\ 4\lambda_2 + 8\lambda_3 \end{bmatrix}$. Solving the resulting linear system, we obtain

$$\begin{bmatrix} 3 & 3 & 4 & 0 \\ 3 & 1 & 0 & 0 \\ 0 & 4 & 8 & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 2 & 4 & 0 \\ 3 & 1 & 0 & 0 \\ 0 & 4 & 8 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1/3 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since there are nonzero solutions, the set is not linearly independent.