Mathematics 3333-002 Name (please print) ____________________

Final Examination Form B

May 11, 2010

Instructions: Give concise answers, but clearly indicate your reasoning.

I. (6) Let $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 3 & 0 \\ -9 & 0 & 0 \end{bmatrix}$.

(a) Calculate the characteristic polynomial of A.

(b) Use the characteristic polynomial to find the only (real) eigenvalue of A.

(c) Find an eigenvector for the eigenvalue.

II. (4) Let $A = ([a_{i,j}])$ be a 5×5 matrix, and consider the formula

$$\det(A) = \sum (\pm) a_{1,\sigma(1)} a_{2,\sigma(2)} a_{3,\sigma(3)} a_{4,\sigma(4)} a_{5,\sigma(5)} .$$

Determine the sign (i. e. tell whether the term has a plus or a minus sign in the formula) of the term that contains $a_{1,3}a_{2,5}a_{3,2}a_{4,4}a_{5,1}$ (make your reasoning clear— answers of "plus" or "minus" without a correct explanation won't receive any credit).

III. (6) Each of the following matrices is the augmented matrix of a system of linear equations, and is in row echelon form or reduced row echelon form. For each matrix, use back substitution to write a general expression for the solutions of the corresponding linear system.

1. $\begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

2. $\begin{bmatrix} 1 & c & b & 1 \\ 0 & 1 & a & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ (the answer will involve a, b, and c)

IV. (4) Let θ be a fixed real number, and in $\mathbb{R}^2$ let $v_\theta = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}$ and $w_\theta = \begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \end{bmatrix}$. Assuming that $\mathbb{R}^2$ has the standard inner product, verify that $\{v_\theta, w_\theta\}$ is an orthonormal basis for $\mathbb{R}^2$.

V. (6) Let $f\colon V \to W$ be a linear transformation between two vector spaces.

(a) Define the *kernel* of f.

(b) Verify that the kernel of f is a subspace of V.

VI. (4) Let V be an inner product space, that is, a vector space V equipped with an inner product denoted by (u, v). Let v_0 be a fixed vector, and let $W = \{v \in V \mid (v, v_0) = 0\}$ be the set of vectors in V that are orthogonal to v_0. Verify that W is a subspace of V.

VII. (15) Label each of the following statements either T for true or F for false. The symbol V that appears in some of the statements denotes a finite-dimensional vector space, and $\{v_1, \ldots, v_k\}$ denotes a finite subset of V.

_______ If $\{v_1, \ldots, v_k\}$ spans V, then some subset of $\{v_1, \ldots, v_k\}$ is a basis for V.

_______ When a homogeneous linear system is written in matrix form $AX = 0$, the rank of A equals the dimension of the solution space of the linear system.

_______ Coordinate vectors satisfy the formulas $(v + w)_S = v_S + w_S$ and $(\lambda v)_S = \lambda v_S$.

_______ If $\{v_1, \ldots, v_k\}$ is a basis for V, and two linear transformation f and g from V to W satisfy $f(v_i) = g(v_i)$ for $1 \leq i \leq k$, then $f(v) = g(v)$ for every v in V.

_______ When a homogeneous linear system is written in matrix form $AX = 0$, the null space of A equals the solution space of the linear system.

_______ If $(\_, \_)$ is an inner product on $\mathbb{R}^n$, and $c_{ij} = (e_i, e_j)$, then the matrix $C = [c_{ij}]$ satisfies $(v, w) = v^T C w$ for any two vectors v and w in $\mathbb{R}^n$.

_______ If a 6×6 matrix has 6 distinct eigenvalues, then it must be diagonalizable.

_______ Similar matrices must have the same characteristic polynomial.

_______ If a matrix is in row echelon form, then its nonzero rows are linearly independent.

_______ If a matrix is in row echelon form, then its nonzero columns are linearly independent.

_______ The only matrix that is similar to the identity matrix is the identity matrix.

_______ A linear transformation is diagonalizable exactly when there is a basis consisting entirely of eigenvectors.

_______ A matrix that has no eigenvectors must be singular.

_______ When a matrix $[a_{i,j}]$ is lower triangular, its characteristic polynomial is $(\lambda - a_{1,1})(\lambda - a_{2,2}) \cdots (\lambda - a_{n,n})$.

_______ The range of a matrix transformation equals the row space of the matrix.

VIII. (4) Recall that if $\{v_1, \ldots, v_k\}$ is a subset of a vector space V, $\operatorname{span}(\{v_1, \ldots, v_k\})$ is the set of linear combinations $\{\lambda_1 v_1 + \cdots + \lambda_k v_k \mid \text{ the } \lambda_i \text{ are numbers.}\}$. Verify that $\operatorname{span}(\{v_1, \ldots, v_k\})$ is a subspace of V.

IX. (9) A certain 3×3 matrix A has eigenvalues 3, -1, and 2.

A 3-eigenvector of A is $\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$, a (-1)-eigenvector of A is $\begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$, and a 2-eigenvector of A is $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

(a) Calculate $A\left(\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} - \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}\right)$.

(b) Calculate $A\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

(c) Tell a 3×3 matrix P such that $P^{-1}AP$ is a diagonal matrix, and tell the diagonal matrix.

X. (10) Let V be the 2-dimensional vector space consisting of solutions to the differential equation $y'' = y$. Recall that e^x, e^{-x}, $\cosh(x) = (e^x + e^{-x})/2$ and $\sinh(x) = (e^x - e^{-x})/2$ are well-known solutions of this equation. Let $S = \{e^x, e^{-x}\}$ and $T = \{\cosh(x), \sinh(x)\}$. These are bases of V (you do not need to verify this).

(a) Find $(3e^x - 2e^{-x})_S$ and $(3e^x - 2e^{-x})_T$.

(b) Find the transition matrix $P_{S\leftarrow T}$, and verify that $P_{S\leftarrow T}(3e^x - 2e^{-x})_T = (3e^x - 2e^{-x})_S$.

(c) Let $D\colon V \to V$ be differentiation, $D(y) = y'$, which is a linear transformation. Find the representation matrix of D with respect to the basis S, and with respect to the basis T.

XI. (6) (a) Tell two elementary 3×3 matrices E and F such that $EFA = B$, where $A = \begin{bmatrix} 2 & -2 & 4 \\ 4 & 1 & -5 \\ 1 & 0 & -3 \end{bmatrix}$ and

$B = \begin{bmatrix} 1 & -1 & 2 \\ 4 & 1 & -5 \\ 4 & -3 & 3 \end{bmatrix}$

(b) Is the matrix $\begin{bmatrix} 2 & -1 & 5 \\ 2 & 0 & 1 \\ 2 & 1 & -3 \end{bmatrix}$ a product of elementary matrices? Why or why not?

XII. (a) Let $\{v_1, \ldots, v_k\}$ be a set of vectors in a vector space V. Define what it means to say that $\{v_1, \ldots, v_k\}$
(6) is linearly independent.

(b) Test $\left\{ \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 8 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix} \right\}$ for linear independence.