Mathematics 3333-002 Name (please print) ____________________

Examination I Form A

February 19, 2010

Instructions: Give concise answers, but clearly indicate your reasoning. Most of the problems have rather short answers, so if you find yourself involved in a lengthy calculation, it might be a good idea to move on and come back to that problem if you have time.

I. (4) Given that $A^{-1} = \begin{bmatrix} 3 & 1 \\ 2 & 0 \end{bmatrix}$ and $B^{-1} = \begin{bmatrix} -2 & 1 \\ 0 & 1 \end{bmatrix}$, find the inverse of AB.

$$(AB)^{-1} = B^{-1}A^{-1} = \begin{bmatrix} -2 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} -4 & -2 \\ 2 & 0 \end{bmatrix}$$

II. (5) Use the row operation method to find the inverse of the matrix $\begin{bmatrix} 3 & 1 \\ a & 0 \end{bmatrix}$ when $a \neq 0$ (the expression for the inverse will involve a in some way).

$$\begin{bmatrix} 3 & 1 & 1 & 0 \\ a & 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 3 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1/a \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 1 & 1 & -3/a \\ 1 & 0 & 0 & 1/a \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 1/a \\ 0 & 1 & 1 & -3/a \end{bmatrix}$$

so the inverse is $\begin{bmatrix} 0 & 1/a \\ 1 & -3/a \end{bmatrix}$. (To check, we can compute that $\begin{bmatrix} 3 & 1 \\ a & 0 \end{bmatrix} \begin{bmatrix} 0 & 1/a \\ 1 & -3/a \end{bmatrix} = I_2$.)

III. (4) Write a system of linear equations in x and y whose solutions, if any, will satisfy

$$\begin{bmatrix} 3 & 1 \\ 2 & 0 \end{bmatrix} = x \begin{bmatrix} 1 & -1 \\ 4 & 0 \end{bmatrix} + y \begin{bmatrix} -1 & 2 \\ 3 & 0 \end{bmatrix}$$

but *do not solve the system* or try to find x and y.

The equation says

$$\begin{bmatrix} 3 & 1 \\ 2 & 0 \end{bmatrix} = x \begin{bmatrix} 1 & -1 \\ 4 & 0 \end{bmatrix} + y \begin{bmatrix} -1 & 2 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} x-y & -x+2y \\ 4x+3y & 0 \end{bmatrix}$$

so a system would be

$$\begin{aligned} x - y &= 3 \\ -x + 2y &= 1 \\ 4x + 3y &= 2 \end{aligned}$$

IV. (4) Write any 3×3 matrix A with the property that

$$A \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 4 \end{bmatrix}$$

Multiplying A by $\begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$ gives twice the middle column of A, so any matrix whose middle column is $(1/2) \begin{bmatrix} -2 \\ 0 \\ 4 \end{bmatrix}$ will work, such as $A = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}$.

V. (5) Find a condition on a and b which tells whether $\begin{bmatrix} a \\ b \end{bmatrix}$ is in the range of the matrix transformation

$$f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2 & 2 & 1 \\ 4 & 4 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} .$$

We investigate when we can solve $\begin{bmatrix} 2 & 2 & 1 \\ 4 & 4 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$. This is the linear system

$$\begin{aligned} 2x + 2y + z &= a \\ 4x + 4y + z &= b \, . \end{aligned}$$

Using Gauss elimination,

$$\begin{bmatrix} 2 & 2 & 1 & a \\ 4 & 4 & 2 & b \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & 2 & 1 & a \\ 0 & 0 & 0 & b-2a \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1/2 & a/2 \\ 0 & 0 & 0 & b-2a \end{bmatrix} ,$$

so the condition to have a solution is that $b = 2a$.

VI. Let A be an $m \times n$ matrix and consider a linear system $AX = B$, where X and B are vectors.
(9)

(a) What must be the dimensions of X and B, when they are regarded as matrices?

X must be $n \times 1$, and B must be $m \times 1$.

(b) Tell the *associated homogeneous system* of $AX = B$.

$AX = 0$.

(c) Suppose that the system has at least one solution X_P. Show that if X_H is any solution of the associated homogeneous system, then $X_P + X_H$ is a solution of $AX = B$.

$A(X_P + X_H) = AX_P + AX_H = B + 0 = B$.

(d) Still assuming that the system has at least one solution X_P, show that *every* solution X of $AX = B$ is of the form $X_P + X_H$ for some solution X_H of the associated homogeneous system. [Hint: start by letting X be any solution. Then, what can you say about $X - X_P$?]

Let X be any solution. Then, $A(X - X_P) = AX - AX_P = B - B = 0$, so $X - X_P$ is a solution of the associated homogeneous solution, and $X = X_P + (X - X_P)$.

VII. Find two elementary matrices E_1 and E_2 so that $E_2E_1A = I_3$, where $A = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix}$.
(6)

The row operations $R_1 - R_3 \to R_1$ and then $R_3 \to (-1/3)R_3$ change A to I_3, so

$$E_1 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1/3 \end{bmatrix}$$

Another possibility is

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1/3 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

VIII. Each of the following matrices is the augmented matrix of a system of linear equations, and is in row echelon
(8) form or reduced row echelon form. For each matrix, use back substitution to write a general expression for the solutions of the corresponding linear system.

1. $\begin{bmatrix} 0 & 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

x_1, x_2, x_4, and x_5 are free parameters, and the first line says that $x_3 = 1 - 2x_4 - x_5$, so the general solution is $(p, q, 1 - 2r - s, r, s)$.

2. $\begin{bmatrix} 1 & a & 0 & b \\ 0 & 1 & c & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ (the answer will involve a, b, and c)

x_3 is a free parameter, the second equation says that $x_2 = -cx_3$, and the first says that $x_1 = b - ax_2 = b - a(-cx_3) = b + acx_3$, so the general solution is $(b + acr, -cr, r)$.

IX. Give an example of two nonzero singular matrices A and B for which $A + B$ is nonsingular.
(3)

There are many examples, perhaps the simplest is

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = I_2 \ .$$

X. Say you have $n \times n$ matrices A and B, and AB is nonsingular. Show that A must also be nonsingular. (That
(5) is, you can't *multiply* two singular matrices and get a nonsingular one.) (If you know about determinants, don't use them here. The solution must only use ideas that we have already studied.)

Say you have AB nonsingular, so there is a C with $ABC = I$. This says that $A(BC) = I$, so BC would be the inverse of A, that is, A must also be nonsingular.