Mathematics 3333-002 Name (please print) ____________________

Examination III Form B

April 28, 2010

Instructions: Give concise answers, but clearly indicate your reasoning. Most of the problems have rather short answers, so if you find yourself involved in a lengthy calculation, it might be a good idea to move on and come back to that problem if you have time.

I. Let A be the matrix
(6)

$$\begin{bmatrix} t & -2 & 0 & 3 \\ 0 & 3 & 1 & 2 \\ 0 & t & 0 & 2 \\ t & 0 & 3 & 4 \end{bmatrix}.$$

(a) Calculate $\det(A)$ as follows. First do an elementary row operation to make the $(4,1)$-entry equal to 0, then do cofactor expansion down the first column to reduce to computing the determinant of a 3×3 matrix. On that 3×3 matrix, do an elementary row operation that creates a second 0 in the middle column, and continue from there.

(b) Using your expression for $\det(A)$, find the values of t for which A is singular.

II. Let $L\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by
(10)

$$L\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \begin{bmatrix} a+2b \\ a-b-c \\ 3a-2c \end{bmatrix}.$$

(You do not need to verify that L is linear.) As you know, the standard matrix representation of L is

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & -1 & -1 \\ 3 & 0 & -2 \end{bmatrix}.$$

(a) Use the standard matrix representation to find a basis for the kernel of L.

(b) Use the standard matrix representation to find a basis for the range of L.

III. (a) Let A be an $n \times m$ matrix, and let $L\colon \mathbb{R}^n \to \mathbb{R}^m$ be the matrix transformation defined by $L(v) = Av$.
(8) Verify that L is linear.

(b) Let P_3 be the space of polynomials of degree at most 3, and let $L\colon P_3 \to P_3$ be the function defined by $L(p(t)) = p(t) + t$. By giving a specific counterexample, show that L is not linear.

IV. (10) Let P_2 be the space of polynomials of degree at most 2, and let S be the ordered basis $\{t^2-t+1, t-1, t^2+1\}$ of P_2.

(a) If the S-coordinate vector of the polynomial p is $p_S = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$, find p.

(b) Find the S-coordinate vector of the polynomial $t^2 + t - 1$.

(c) Let T be the basis $\{t^2, t, 1\}$ of P_2. Find the transition matrix (also called the change-of-basis matrix) $P_{T \leftarrow S}$ from S-coordinates to T-coordinates.

V. (6) Let P be a nonsingular $n \times n$ matrix.

(a) Verify that $\det(P^{-1}) = 1/\det(P)$.

(b) Use part (a) to verify that if A is any $n \times n$ matrix, then $\det(P^{-1}AP) = \det(A)$.

VI. (4) Let $A = ([a_{i,j}])$ be a 4×4 matrix, and consider the formula

$$\det(A) = \sum (\pm) a_{1,\sigma(1)} a_{2,\sigma(2)} a_{3,\sigma(3)} a_{4,\sigma(4)} \ .$$

Determine the sign (i. e. tell whether the term has a plus or a minus sign in the formula) of the term that contains $a_{1,2}a_{2,3}a_{3,4}a_{4,1}$ (make your reasoning clear— answers of "plus" or "minus" without a correct explanation won't receive any credit).

VII. (5) Let V be a vector space of dimension 3, and let $T = \{t_1, t_2, t_3\}$ be an ordered basis of V. Let $L\colon V \to V$ be the linear transformation whose matrix representation with respect to T-coordinates on the domain and the codomain is $A = \begin{bmatrix} 0 & -2 & 1 \\ 1 & 2 & 1 \\ 0 & 2 & 0 \end{bmatrix}$. Write $L(2t_1 + t_2 - t_3)$ as a linear combination of t_1, t_2, and t_3.

VIII. (4) An $n \times n$ matrix B is obtained from a matrix $A = [a_{i,j}]$ by the elementary row operation $kR_i \to R_i$. Use the formula for $\det(A)$ to explain why $\det(B) = k\det(A)$.