A Topology is the minimal amount of structure needed to have a workable def. of Continuity

Def'n: let X be a Set.
A topology on X is a collection

$\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ of subsets of X

satisfying — called the open subsets for the topology

(T1) $X \in \mathcal{U}$
and $\emptyset \in \mathcal{U}$

(T2) if $\{U_\beta\}_{\beta \in B} \subseteq \mathcal{U}$, then $\bigcup_{\beta \in B} U_\beta \in \mathcal{U}$

(a union of open sets is open)

(T3) If $\{U_i\}_{i=1}^{n} \subseteq \underset{\text{the top.}}{\mathcal{U}}$, then $\bigcap_{i=1}^{n} U_i \in \mathcal{U}$

(an intersection of finitely many open sets is open)

(T2 - T3)
i.e. $\mathcal{U}$ is closed under finite intersection and arbitrary union

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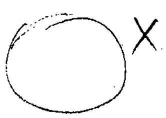
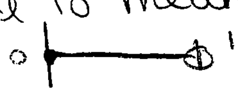


$P(X) = \{\emptyset, \{x\}, \{y\}, \dots\}$ ← all subsets of X
$\cup$I
$\mathcal{U}$

Ex (1.) $X = \mathbb{R}^n$, $\mathcal{U} = \{U \subseteq \mathbb{R}^n \mid U \text{ is a union of open balls}\} \cup \{\emptyset\}$
This is the standard Topology
(T1) $\mathbb{R}^n = \bigcup_{x \in \mathbb{R}^n} B(x,1)$ $\quad \emptyset = \bigcup$

a "(" here would give the standard top

(2.) $X = \mathbb{R}$, $\mathcal{U} = \{U \subseteq \mathbb{R} \mid \forall x \in U, \exists a, b \in \mathbb{R} \text{ s.t. } x \in [a,b) \subseteq U\}$
Called the lower limit topology
We can write $\mathbb{R}_\ell$ to mean $\mathbb{R}$ with
In $\mathbb{R}_\ell$, $[0,1)$ is open



Pf: let $x \in [0,1)$. Then $x \in [x,1) \subseteq [0,1)$. $\square$

- In $\mathbb{R}_\ell$, $(0,1]$ is not open because if $1 \in [a,b)$, then $1 < b$, so $[a,b) \not\subseteq (0,1]$.

- $(0,1)$? is open in $\mathbb{R}_\ell$
let $x \in (0,1)$, then $[x,1) \subseteq (0,1)$

- Similarly, any $[r,s)$ and any (r,s) are open.
- Any set open in the standard topology is open in $\mathbb{R}_\ell$

(3.) A small topology:

$X = \mathbb{R}$, $\mathcal{U} = \{U \subseteq \mathbb{R} \mid \mathbb{R} - U \text{ is finite}\} \cup \{\emptyset\}$
This is called the <u>cofinite</u> topology on $\mathbb{R}$

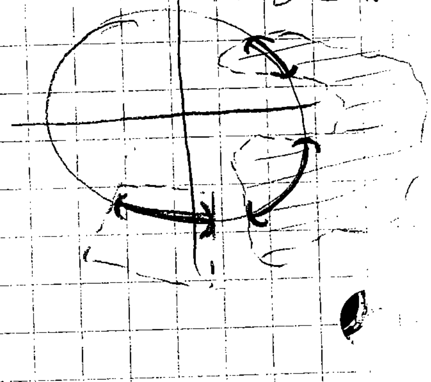
very few open sets

(T1) $\mathbb{R} - \mathbb{R} = \emptyset$, so $\mathbb{R} \in \mathcal{U}$, & $\emptyset \in \mathbb{R}$
(T2) let $\{U_\beta\}$ be open sets
If all $U_\beta = \emptyset$, then $\bigcup U_\beta = \emptyset \in \mathcal{U}$
Suppose $U_{\beta_0} \neq \emptyset$, so $\mathbb{R} - U_{\beta_0}$ is finite
$\mathbb{R} - \bigcup U_\beta = \bigcap(\mathbb{R} - U_\beta) \subseteq \mathbb{R} - U_{\beta_0}$, which is finite
$\therefore \bigcup U_\beta$ is open.

(4.) X is any set $\{X, \emptyset\}$, the <u>indiscrete topology</u> on X

(5.) X is any set $\mathcal{U} = \{U \mid U \subseteq X\}$ is the <u>discrete</u> top.
- every point is far away from ea. other

$X = S^1 \subseteq \mathbb{R}^2$



(6.) X = a subset of $\mathbb{R}^n$
define $\mathcal{U} = \{U \cap X \mid U \text{ is open in } \mathbb{R}^n\}$
called the <u>Subspace Top.</u> on X

Verify:

(T1) $\emptyset = \underbrace{\emptyset}_{\text{open in } \mathbb{R}^2} \cap S^1$, $S^1 = \underbrace{\mathbb{R}^2}_{\text{open in } \mathbb{R}^2} \cap S^1$
(T2) let $\{V\}_{\alpha \in A} \in \mathcal{U}$ means ea. $V_\alpha \in \mathcal{U}$,
so $\exists\, U_\alpha$ open in $\mathbb{R}^2$ w/ $V_\alpha = U_\alpha \cap S^1$
$\bigcup V_\alpha = \bigcup(U_\alpha \cap S^1) = \underbrace{(\bigcup U_\alpha)}_{\text{open in } \mathbb{R}^2} \cap S^1 \in \mathcal{U}$
(T3) Similar to T2
* This works for any X in place of S^1 & any space in place of $\mathbb{R}^2$

(T3) If $\underbrace{\{U_i\}_{i=1}^{n} \subseteq \mathcal{U}}_{\text{every } U_i \in \mathcal{U}}$, then $\bigcap_{i=1}^{n} U_i \in \mathcal{U}$

on $\mathbb{R}$, ($\mathbb{R}_\ell$) the lower limit topology
$[0,1)$ is open.

NOTE: (T3) could be replaced by (T3'):
If $U, V \in \mathcal{U}$, then $U \cap V \in \mathcal{U}$ (intersection of finitely many sets is open)

(T3) $\Rightarrow$ (T3')

(T3') $\Rightarrow$ (T3): Assume (T3')
Induct on n, starting at $n=2$ ($=$T3')
Assume T3 is true for $n-1$
Let $U_1, \ldots U_n \in \mathcal{U}$. $\bigcap_{i=1}^{n} U_i = \underbrace{\left(\bigcap_{i=1}^{n-1} U_i\right)}_{\text{open by induction}} \cap U_n$ is open by (T3')

$\therefore$ By induction, (T3) is true for all n. ∎

So to verify $\mathcal{U}$ is a Topology, you only need to prove (T3')

Note: Ex. 6 is: let Y be any topological Space, let $X \subseteq Y$. The Subspace top. on X is
$\mathcal{U} = \{U \cap X \mid U \text{ open in } Y\}$
$\rightarrow$ gives many Examples

Verify $\mathbb{R}_\ell$ is a topology:

$\mathcal{U} = \{ U \mid \forall x \in U, \exists a, b \in \mathbb{R} \text{ s.t. } x \in [a,b) \subseteq U \}$

(T1) $\emptyset$: $\forall x \in \emptyset, \ldots$ is "vacuously" true.
$\mathbb{R}$: let $x \in \mathbb{R}$, $x \in [x, x+1) \subseteq \mathbb{R}$

(T3) Suppose $U_1, \ldots U_n \in \mathcal{U}$
let $x \in \bigcap_{i=1}^{n} U_i$

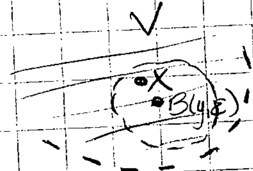
For each i, $x \in U_i$ & U_i is open
$\exists a_i, b_i$ w/ $x \in [a_i, b_i) \subseteq U_i$

let $a = \max\{a_i\}$, $b = \min\{b_i\}$
$a \le x$ and $x < b$, so $x \in [a,b)$
For each i, $[a,b) \subseteq [a_i, b_i)$
so, $[a,b) \subseteq \bigcap_{i=1}^{n} [a_i, b_i) \subseteq \bigcap_{i=1}^{n} U_i$

(T2) exercise

U is open in $\mathbb{R}^n$ when $\forall x \in U$, $\exists \varepsilon > 0$, $x \in B(x, \varepsilon) \subseteq U$

Could we define open sets as:
(*) $\forall x \in U$, $\exists$ an open ball w/ $x \in B \subseteq U$
1.) Any set open in the standard def. would be open according to (*)
2.) Suppose V satisfies (*)



Let $x \in V$.
Let B be an open ball with $x \in B \subseteq V$
$B = B(y, \varepsilon)$ for some $y \in V$ and $\varepsilon > 0$
Then $B(x, \varepsilon - \|x - y\|) \subseteq B(y, \varepsilon) \subseteq V$
So B is open in the standard sense

(*) implies every open set is a union of open balls

Generalize This:

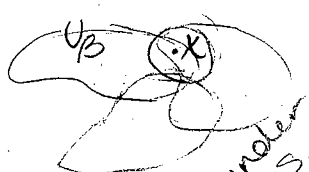
- let X be a set, & let $\mathcal{B}$ be a collection of Subsets of X.
  (e.g. $\mathbb{R}^n$ and $\{B(x,\varepsilon)\}$
  $\mathbb{R}$ and $\{[a,b)\}$

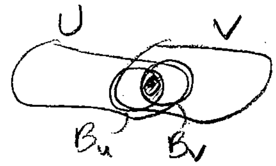
define $\mathcal{U} = \{U \subseteq X \mid \forall x \in U, \exists B \in \mathcal{B}, x \in B \subseteq U\}$
Note: ($U \in \mathcal{U}$ iff U is a union of elmts of $\mathcal{B}$)

Under what conditions on $\mathcal{B}$ will $\mathcal{U}$ be a topology on X? –

1.) $\emptyset \in \mathcal{U}$ because $\forall x \in \emptyset, \ldots$ is true
For $X \in \mathcal{U}$, we need $\forall x \in X\ \exists B \in \mathcal{B}, x \in B$ $(\subseteq X)$
(all B's are subsets of X)
- An easy way to say this is that
$X = \bigcup_{B \in \mathcal{B}} B$

2.) Let $\{U_\alpha\} \subseteq \mathcal{U}$. let $x \in \bigcup U_\alpha$ so $x \in U_\beta$ for some U_β
$U_\beta \in \mathcal{U}$, so $\exists B \in \mathcal{B}, x \in B \subseteq U_\beta \subseteq \bigcup U_\alpha$
$\therefore \bigcup U_\alpha \in \mathcal{U}$





(T3') 3.) let U and $V \in \mathcal{U}$.
let $x \in U \cap V$
$\exists B_u \in \mathcal{B}, x \in B_u \subseteq U$
$\exists B_v \in \mathcal{B}, x \in B_v \subseteq V$ need $x \in B_u \cap B_v \subseteq U \cap V$ but can't assume $\mathcal{B}$ is closed under taking inters. we want a weaker cor.
[Require if B_1 and $B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$,
There exists some element $\exists B \in \mathcal{B}$ s.t. $x \in B \subseteq U \cap V$]

(HW) 22: $\sqrt{2}$ $\mathbb{Q} = (\mathbb{Q} \cap (-\infty, \sqrt{2})) \cup (\mathbb{Q} \cap (\sqrt{2}, \infty))$
- open, nonempty & disjoint, $\cup = \mathbb{Q}$
($\mathbb{Q}$ is not connected)

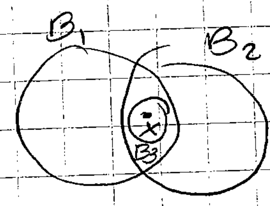
Reals x if Union is the reals, must be overlap
- to cut in 2 pieces you must have one set half closed

$r \notin \mathbb{Q}$ q_1 q_2

any subset of $\mathbb{Q}$ is not connected, take two intervals & check properties

let U be open in $\mathbb{Q}$,
$U = \mathbb{Q} \cap V$ for some open V in $\mathbb{R}$
V contains an open interval (a,b)
Choose $q_1 < q_2$ in $\mathbb{Q} \cap (a,b) \subseteq U$
choose $r \notin \mathbb{Q}$ with $q_1 < r < q_2$ let
$U_1 = (-\infty, r) \cap U$ $\quad U_2 = (r, \infty) \cap U$
· use def. to check properties

<u>def. of Basis</u>
Let X be a set and let $\mathcal{B}$ be a collection of subsets of X. We say $\mathcal{B}$ is a <u>basis</u> if



(B1) $X = \bigcup_{B \in \mathcal{B}} B$

(B2) if $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$,
$\exists B_3 \in \mathcal{B}$, $x \in B_3 \subseteq B_1 \cap B_2$

- If $\mathcal{B}$ is a basis, the collection
  $\mathcal{U} = \{ U \subseteq X \mid \forall x \in U, \exists B \in \mathcal{B}, x \in B \subseteq U \}$
  is a topology on X.
- The sets in $\mathcal{B}$ are open and are called <u>basic open sets</u>.
- $\mathcal{U}$ is a <u>topology generated by $\mathcal{B}$</u>
- $\mathcal{B}$ is a basis for $\mathcal{U}$
- The sets in $\mathcal{U}$ are exactly the union of subcollections of $\mathcal{B}$

Ex let X be a set, $\mathcal{B}$ a collection of subsets of X <u>whose union is X</u> and which is closed under finite intersections
(if $B_1, B_2 \in \mathcal{B}$, then $B_1 \cap B_2 \in \mathcal{B}$)

Then $\mathcal{B}$ is a basis: (B1) given by hyp.
(B2) suppose $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$
$B_1 \cap B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2 \subseteq B_1 \cap B_2$

- For example, any topology $\mathcal{U}$ is a basis (that generates itself)

$\boxed{Ex^2}$ $X = \mathbb{R}$ and $\mathcal{B} = \{[a,b) \mid a,b \in \mathbb{R}\}$ is a basis for $\mathbb{R}_\ell$

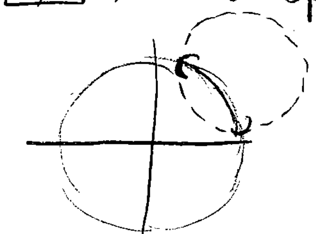
$\boxed{Ex^3}$ for the standard topology on $\mathbb{R}^n$, The following are bases:

1.) $\{B(x,\varepsilon) \mid x \in \mathbb{R}^n, \varepsilon > 0\}$
2.) $\{S(x,\varepsilon)\}$ all open cubes
3.) $\{B((r_1,r_2,\dots,r_2), r) \mid$ where $r, r_1, \dots, r_n \in \mathbb{Q}\}$

NOTE: This Basis ↑ is small, it contains only countably many elements
-makes it easier to work w/

4.) $\{(a_1,b_1) \times (a_2,b_2) \times \dots \times (a_n,b_n)\}$ all open rectangles

$\boxed{Ex^4}$ X is a space, $\mathcal{B}$ a basis for the topology, and $A \subseteq X$



$\{B \cap A \mid B \in \mathcal{B}\}$ is a basis for the subspace topology on A

Countability:

An infinite set X is called countable if there is a bijection $\phi: X \to \mathbb{N}$ where $\mathbb{N}$ is the set of natural numbers $\mathbb{N} = \{1,2,\dots\}$
(Sometimes finite sets are considered countable)

$\boxed{Ex}$ $X = \{2,4,6,8,\dots\}$ $\phi: X \to \mathbb{N}$ by $\phi(K) = \frac{K}{2}$

$$\begin{matrix} 2 & 4 & 6 & \dots \\ \downarrow & \downarrow & \downarrow & \\ 1 & 2 & 3 & \dots \end{matrix}$$

$\boxed{Ex^2}$ $X = \{\dots,-3,-2,1,0,1,2,\dots\}$

$$\begin{matrix} 0 & 1 & -1 & 2 & -2 & \dots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ 1 & 2 & 3 & 4 & 5 & \end{matrix}$$

- for Countable sets you can make a list of elements.

- Any infinite subset of a countable set is countable

argument → 1.) Say $A \subseteq X$ and X is countable
Make a list of X & take out things not in A and you have a list of A

$$\begin{array}{cccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \cdots & \phi(a) \\ x_1 & x_2 & x_3 & \cdots & & a_1 & \cdots & & a_2 & a_3 & x_8 \cdots & a \end{array}$$

$[1, \phi(a)]$ $\qquad X \xrightarrow{\phi} \mathbb{N}$

$\phi^{-1}[1, \phi(a)] \xrightarrow{\phi} [1, \phi(a)]$

$\phi^{-1}[1, \phi(a)] \cap A$

Pf. Say $A \subseteq X$, X is countable

let $\phi : X \to \mathbb{N}$ be a bijection (inj. and sur.)

define $\phi_A : A \to \mathbb{N}$

defined by

$\phi_A(a) = \left| \phi^{-1}[1, \phi(a)] \cap A \right|$ ← take cardinality (# of elements)

Prove ϕ_A is injective & surjective...

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25. A $\quad V \quad$ $\mathcal{B}_A = \{B \cap A \mid B \in \mathcal{B}\}$ is a basis for a subspace top

let U be open in subspace Top,

$B \cap A$ $\quad x \in B \cap A \subseteq U$

• An infinite subset of a countable set is countable

Prove it is a bijection:

ϕ_A is <u>injective</u>:

Suppose $a, a' \in A$ and $a \neq a'$

$\phi(a) \neq \phi(a')$ because ϕ is a bijection

[w/out loss of generality] We may assume that $\phi(a) < \phi(a')$

because

① You could rename the elements giving the name a to whichever is smaller ϕ value

② After the argument is finished, you could swap the a and a' wherever they appear

So $[1, \phi(a)] \subseteq [1, \phi(a')]$

$\phi^{-1}([1, \phi(a)]) \cap A \subseteq \phi^{-1}([1, \phi(a')]) \cap A$

$a' \in \phi^{-1}([1, \phi(a')]) \cap A$

But $a' \notin \phi^{-1}([1, \phi(a)]) \cap A$

$\therefore \left| \phi^{-1}([1, \phi(a)]) \cap A \right| < \left| \phi^{-1}([1, \phi(a')]) \cap A \right|$

$\phi_A(a) < \phi_A(a')$

$\mathbb{N}$
$x \quad x \quad x \quad a_1 \quad a_2 \quad a_3 \cdots \quad a_{n-1} [\text{---} a$

Surjective:

let N be the smallest element of $\phi(A)$

$|\phi^{-1}([1, N]) \cap A| = 1$ (b/c only 1 element of A in this)

Let $a_1 = \phi^{-1}([1, N]) \cap A$ so $\phi_A(a_1) = 1$ so $1 \in$ range ϕ_A

Induct:

Assume $n-1$ is in the range of (ϕ_A)

let $a_{n-1} \in A$ with $\phi_A(a_{n-1}) = n-1$

$|\phi^{-1}([1, \phi(a_{n-1})])| \cap A|$ contains $n-1$ elements

$\phi_A(A)$ is infinite, so is not contained in $[1, \phi(a_{n-1})]$

(There must be more than the a_{n-1} element)

- want to find a_n

Let $\phi(a)$ be the smallest element of

$[\phi(a_{n-1})+1, \infty) \cap \phi(A)$

Claim $\phi_A(a) = |\phi^{-1}([1, \phi(a)]) \cap A| = |\phi^{-1}([1, \phi(a_{n-1})] \cup \{a\}) \cap A|$

$= (n-1) + 1 = n$

So $n \in \text{range}(\phi_A)$

- The set of rational numbers is countable

Use the Cantor method: (1870's)

NOTE: $\frac{p_1}{q_1} \sim \frac{p_2}{q_2}$ when $p_1 q_2 = p_2 q_1$ (an equivalence relation)

$[\frac{2}{4}] = [\frac{1}{2}] = \{ \ldots \frac{-2}{-4}, \frac{2}{4}, \ldots \frac{1}{2}, \frac{-1}{-2} \ldots \}$

$$\begin{array}{cccccc} \frac{1}{1} & \frac{2}{1} & \frac{3}{1} & \frac{4}{1} & \frac{5}{1} & \cdots \\ \frac{1}{2} & \frac{2}{2} & \frac{3}{2} & \frac{4}{2} & \frac{5}{2} & \cdots \\ \frac{1}{3} & \frac{2}{3} & \frac{3}{3} & \frac{4}{3} & \frac{5}{3} & \cdots \end{array}$$

diagonal gives them an order

$1, \frac{2}{1}, \frac{1}{2}, \frac{1}{3}, \cancel{\frac{2}{2}}, \frac{3}{1}, \frac{4}{1}, \frac{3}{2}, \frac{2}{3}, \ldots$

- throw out ones not in lowest terms

- Start with zero & insert neg.

$0, 1, -1, \frac{1}{2}, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{3}, 3, -3, 4, -4, \frac{3}{2}, -\frac{3}{2}, \frac{2}{3}, -\frac{2}{3}, \ldots$

- The Reals are Uncountable!!

- 2 countable sets, then there is a bijection between the 2 countable sets

- Every $\mathbb{N}$ is an open set, $\therefore$ Every subset of $\mathbb{N}$ is open b/c it is a union of 1 pt. sets

Uncountable Sets

• $\mathbb{R}$ is uncountable: Cantor diagonal argument:
Suppose for contradiction that $\mathbb{R}$ is countable

$z_1 . a_{11} a_{12} a_{13} a_{14} \ldots$ $z_i \in \mathbb{Z}$ and $a_{ij} \in \{0, 1, \ldots, 9\}$
$z_2 . a_{21} a_{22} a_{23} a_{24} \ldots$
$z_3 . a_{31} a_{32} a_{33} a_{34} \ldots$
$\vdots$

show that it is not a complete list:
look at diagonal
define $b_i = \begin{cases} 6 & \text{if } a_{ii} \neq 6 \\ 8 & \text{if } a_{ii} = 6 \end{cases}$

Then, the number $0.b_1 b_2 b_3 \ldots$
is not on the list for the i^{th} number in the list, $a_{ii} \neq b_i$ so r does not equal the i^{th} number

• if A and B are countable, then their Cartesian product, $A \times B = \{(a,b) \mid a \in A, b \in B\}$ is countable

$A = \{a_1, a_2, a_3, \ldots\}$
$B = \{b_1, b_2, b_3, \ldots\}$

$(a_1, b_1), (a_1, b_2), (a_1, b_3) \ldots$ so all (a_i, b_j) is here
$(a_2, b_1), (a_2, b_2), (a_2, b_3) \ldots$
(a_3, b_1)
$\vdots$

Can make a single list by using the Cantor method to make a complete list of the elements

• if $A_1, \ldots, A_n$ are countable, then their product $A_1 \times A_2 \times \cdots \times A_n$ is countable

Pf: There is a bijection $(A_1 \times A_2 \times \cdots \times A_n) \longleftrightarrow (A_1 \times A_2 \times \cdots \times A_{n-1}) \times A_n$
$(a_1, a_2, \ldots, a_n) \longrightarrow ((a_1, a_2, \ldots a_{n-1}), a_n)$

can assume this is countable b/c Product of count. sets

can assume this is countable by induction

$\boxed{Ex}$ $\{B((r_1, r_2, \ldots, r_n), \frac{1}{m}) \mid r_i \in \mathbb{Q}, m \in \mathbb{N}\} = \mathcal{B}$

This is countable b/c bijection b/two

$\mathbb{Q}^n \times \mathbb{N}$ is a countable set

$\mathbb{Q} \times \mathbb{Q} \times \cdots \mathbb{Q}$

And this $\mathcal{B}$ is a basis for the standard top.

- If $A_1, A_2, \ldots$ are countable sets, then their union, $\bigcup_{n=1}^{\infty} A_n$ is countable

$$\begin{array}{ll} A_1: & a_{11}\ a_{12}\ a_{13} \ldots \\ A_2: & a_{21}\ a_{22}\ a_{23} \ldots \\ A_3: & a_{31}\ a_{32}\ a_{33} \ldots \end{array}$$

(b/c you can make a list of each A)

- Use the Cantor method to make this into a list, and throw out repeats

NOTE: - This works even when some A_i's are finite

$$\begin{array}{ll} A_1: & a_{11}\ a_{12}\ a_{13}\ a_{14}\ a_{15} \ldots \\ A_2: & a_{21}\ a_{22} \\ A_3: & a_{31}\ a_{32}\ a_{33}\ a_{34} \ldots \end{array}$$

(just skip blank places)

⑨ • if A is countable, then the collection of <u>finite</u> subsets of A is countable

<u>Pf</u>: let $\mathcal{F}_n$ be the collection of n-element subsets of A. $\mathcal{F}_0 = \{\emptyset\}$ (has one element)

$\mathcal{F}_1 \longleftrightarrow A$ $\quad$ $\mathcal{F}_2$ • Use induction to show $\mathcal{F}_n$ is count.

$\{a\} \longleftrightarrow a$

Write $A = \{a_1, a_2, \ldots\}$. let $\mathcal{F}_{2,a_i} = \{\{a_i, x\} \mid x \in A - \{a_i\}\}$

$\mathcal{F}_{2,a_i} \longleftrightarrow A - \{a_i\}$

$\{a_i, x\} \longleftrightarrow x$

So $\mathcal{F}_{2,a_i}$ is countable

and $\mathcal{F}_2 = \bigcup_{i=1}^{\infty} \mathcal{F}_{2,a_i}$ so is countable

March 1: ... Pf. Continued:

Write $A = \{a_1, a_2, a_3, \dots\}$
For each $x \in A$, define $\mathcal{F}_{2,x} = \{\{a,x\} \mid a \in A - \{x\}\} \longleftrightarrow A - \{x\}$
$\mathcal{F}_{2,x}$ is countable $\qquad \{a,x\} \longleftrightarrow \{a\}$

$\mathcal{F}_2 = \bigcup_{x \in A} \mathcal{F}_{2,x}$ is a union of countably many sets, so it is countable.

Inductively, Assume $\mathcal{F}_{n-1}$ is countable. for (n-1)element subsets of any countable set.
For each x define $\mathcal{F}_{n,x} = \{\{x, y_1, y_2, \dots y_{n-1}\} \mid$
$\{y_1, y_2, \dots y_{n-1}\}$ is an (n-1)element subset of $A - \{x\}\}$

$\mathcal{F}_{n,x} \longleftrightarrow$ the (n-1)-element subsets of countable set $A - \{x\}$.
So $\mathcal{F}_{n,x}$ is countable.

$\mathcal{F}_n = \bigcup_{x \in A} \mathcal{F}_{n,x}$ so is countable.

$\bigcup_{n=0}^{\infty} \mathcal{F}_n$ is the collection of all finite subsets of A and is countable.

(10) • If A is countable, then the collection $P(A)$ of all subsets of A (The Power set of A) is uncountable.

Pf: Suppose for Contradiction, that $P(A)$ is countable. Then there is a bijection from A to $P(A)$, $\phi : A \to P(A)$
$a \mapsto \phi(a)$

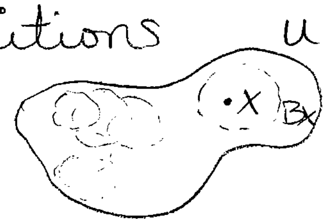
define $B \subseteq A$ by $B = \{a \mid a \notin \phi(a)\}$ $\quad B \in P(A)$
Show that B is not in the range, so ϕ isn't surjecti
Suppose $\phi(a_0) = B$
Case I: $a_0 \in B$. Since $B = \phi(a_0)$, we have $a_0 \in \phi(a_0)$, so $a_0 \notin B$ $\rightarrow\!\leftarrow$
Case II: $a_0 \notin B$. Since $B = \phi(a_0)$, $a_0 \notin \phi(a_0)$
$\therefore a_0 \in B$ $\rightarrow\!\leftarrow$
So $\phi(a_0) = B$ is impossible. ∎

(11) • if A is any set, there is no surjective function from $A \to P(A)$
(Same argument a (10))

larger sets of infinity: (no surjection)
$\mathbb{N} \ldots P(\mathbb{N}) \ldots P(P(\mathbb{N}))$

Back to Bases!

<u>Problem</u>: let X be a topological space, with its topology $\mathcal{U}$.
How can we recognize when a collection, $\mathcal{B}$, of subsets of X is a Basis <u>for $\mathcal{U}$</u>?
We want:
(a) every open set in X is a union of elements of $\mathcal{B}$
and (b) Every union of elements of $\mathcal{B}$ is an open set in X.
and (c) $\mathcal{B}$ should satisfy the conditions to be a basis.



(a) will be achieved if
$\forall U \in \mathcal{U}, \forall x \in U, \exists B \in \mathcal{B}, x \in B \subseteq U$ ← (1)
i.e. $\mathcal{U}$ is in the top. generated by $\mathcal{B}$

(b) will be assured if every element of $\mathcal{B}$ is open in X ← (2)

assuming

Try to prove that $\mathcal{B}$ is a basis, and see what else we need:
(B1) let $x \in X$. X is open.
by (1), $\exists B \in \mathcal{B}$ s.t. $x \in B \subseteq X$
$\therefore X = \bigcup_{x \in X} \{x\} \subseteq \bigcup_{B \in \mathcal{B}} B \subseteq X$

(B2) Let $B_1, B_2 \in \mathcal{B}$ and let $x \in B_1 \cap B_2$
by (2), B_1 and B_2 are open, so $B_1 \cap B_2$ is open
by (1), $\exists B \in \mathcal{B}$ s.t. $x \in B \subseteq B_1 \cap B_2$

Basis Recognition Thm:
Let X be a topological Space.
Let $\mathcal{B}$ be a collection of open subsets s.t. $\forall U$ open in X, $\forall x \in U$, $\exists B \in \mathcal{B}$, $x \in B \subseteq U$
Then $\mathcal{B}$ is a basis that generates the top. on X.