Basis Recognition Thm: a.k.a. let $(X, \mathcal{U})$ be a topology

- Let X be a topological Space.
  Let $\mathcal{B}$ be a collection of open subsets ①
  s.t. $\forall U$ open in X, $\forall x \in U$, $\exists B \in \mathcal{B}$, $x \in B \subseteq U$ ②
- Then $\mathcal{B}$ is a basis that generates the top. on X.

[a]

#29 HW: if $B_1, B_2 \in \mathcal{B}$, then they are open $\Rightarrow B_1 \cap B_2$ is open
if know it's a basis, it generates some topology
[b] let $\mathcal{U}_{\mathcal{B}}$ be the top. generated by $\mathcal{B}$

$\boxed{\text{March 3:}}$

Ex let $\mathcal{B}_{rational} = \{ B((r_1, r_2, \ldots, r_n), \frac{1}{m}) \mid r_i \in \mathbb{Q}, m \in \mathbb{N} \}$

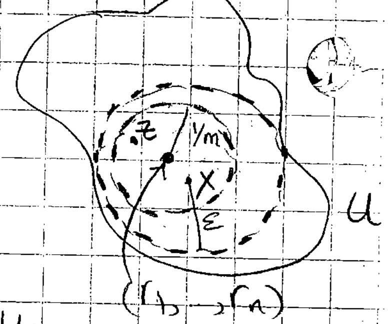
$\Rightarrow$ countable $\quad \updownarrow \; \mathbb{Q}^n \times \mathbb{N}$

We will use the Basis Recognition Thm. to prove $\mathcal{B}_{rational}$ is a basis for the standard topology:

1.) The sets in $\mathcal{B}_{rational}$ are open (b/c they're open balls in the standard topology

2.) let U be open and $x \in U$

def. of standard topology $\rightarrow$ Since U is open, $\exists \varepsilon > 0$ s.t. $B(x, \varepsilon) \subseteq U$



- Choose a positive integer m with $1/m < \frac{\varepsilon}{2}$

Write $x = (x_1, \ldots, x_n)$
for each i, choose a rational r_i with
$x_i < r_i < x_i + 1/m\sqrt{n}$

$$\|x - (r_1, \ldots, r_n)\| = \left(\sum_{i=1}^{n} (x_i - r_i)^2\right)^{1/2} < \left(\sum_{i=1}^{n} \left(\frac{1}{m\sqrt{n}}\right)^2\right)^{1/2}$$

$$= \left(\sum_{i=1}^{n} \frac{1}{m^2 n}\right)^{1/2} = \left(\frac{1}{m^2}\right)^{1/2} = \frac{1}{m}$$

$$\underbrace{\frac{1}{n}\left(\frac{1}{m^2}\right) + \frac{1}{n}\left(\frac{1}{m^2}\right) + \ldots + \frac{1}{n}\left(\frac{1}{m^2}\right)}_{n \text{ times}} = \frac{1}{m^2}$$

So $x \in B((r_1, \ldots, r_n), \frac{1}{m})$

Let $z \in B((r_1, \ldots, r_n), \frac{1}{m})$.

$$\|z - x\| = \|z - (r_1, \ldots, r_n)\| + \|(r_1, \ldots, r_n) - x\|$$
$$< \frac{1}{m} + \frac{1}{m} < \varepsilon$$

$\therefore B((r_1, \ldots, r_n), \frac{1}{m}) \subseteq B(x, \varepsilon) \subseteq U$

$\Rightarrow$ The Standard top on $\mathbb{R}^n$ has a countable Basis

NOTE:
- Can Use Basis Rec. Thm. for HW #25 also

Continuity:

let $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ be 2 top. Spaces and let $f: X \to Y$

def: f is continuous if for every $V \in \mathcal{V}$, $f^{-1}(V) \in \mathcal{U}$

i.e. the inverse image of every open set is open (useful def. for spaces w/out def. of distance such as the cofinite top.)

Examples:

(1.) We already proved that $f: \mathbb{R}^m \to \mathbb{R}^n$ is continuous in the ε-δ def. iff it is continuous in the topological def.

an extreme example (2.) let X be a set and $\mathcal{D}$, the discrete top. on X. Let $(Y,\mathcal{V})$ be any space and $f:(X,\mathcal{D}) \to (Y,\mathcal{V})$ be any function, Then f is continuous.

$\boxed{Ex}$' $\mathbb{N} \subseteq \mathbb{R}$, the subspace topology on $\mathbb{N}$ is the discrete topology.

check: let $n \in \mathbb{N}$

$(n-\frac{1}{2}, n+\frac{1}{2}) \cap \mathbb{N} = \{n\}$

$\Rightarrow$ each $\{n\}$ is open in the Subspace top.

If $U \subseteq \mathbb{N}$, the $U = \bigcup_{n \in U} \{n\}$, so U is open in the Subspace topology

$\boxed{Ex}^2$

note: A function $f: \mathbb{N} \to \mathbb{R}$ is called a sequence, so all sequences are continuous

(3.) Let $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ be spaces (with $Y \neq \emptyset$)

Let $y_0 \in Y$, define $C_{y_0}: (X,\mathcal{U}) \to (Y,\mathcal{V})$ by

$C_{y_0}(x) = y_0$ for all x (The constant function)

(all of X goes to the single point)

c_{y_0} is always continuous:

Pf: Let V be open in Y.

$$c_{y_0}^{-1}(V) = \begin{cases} X & \text{if } y_0 \in V \\ \emptyset & \text{if } y_0 \notin V \end{cases}$$

Since X and $\emptyset$ are open, c_{y_0} is continuous.

(4.) Compositions of continuous functions are continuous

HW: look at $(g \circ f)^{-1}(W) = f^{-1}(g^{-1}(W))$

check: let $x \in (g \circ f)^{-1}(W)$

means $(g \circ f)(x) \in W$

$\Rightarrow g(f(x)) \in W \Rightarrow f(x) \in g^{-1}(W)$

$\Rightarrow x \in f^{-1}(g^{-1}(W))$

Let $x \in f^{-1}(g^{-1}(W))$...

(5.) If $\mathcal{B}$ is a basis for the topology on Y, and $f^{-1}(B)$ is open for every $B \in \mathcal{B}$, Then f is continuous.

$\boxed{\text{March 5}}$:

Facts: • $f^{-1}(\bigcup A_\alpha) = \bigcup f^{-1}(A_\alpha)$

$x \in f^{-1}(\bigcup A_\alpha) \Leftrightarrow f(x) = \bigcup A_\alpha$

$\Leftrightarrow f(x) \in A_{\alpha_0}$ for some α_0

$\Leftrightarrow x \in f^{-1}(A_{\alpha_0})$ for some α_0

$\Leftrightarrow x \in \bigcup f^{-1}(A_\alpha)$

• $f^{-1}(\bigcap A_\alpha) = \bigcap f^{-1}(A_\alpha)$

BUT Careful when taking images, for example $f(A \cap B)$ may not equal $f(A) \cap f(B)$

• $f(\bigcup A_\alpha) = \bigcup f(A_\alpha)$ (but doesn't work w/ $\bigcap$)

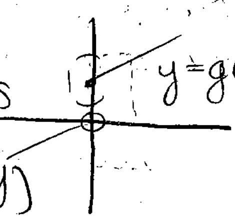
(6.) let $\mathcal{S}$ = standard top. on $\mathbb{R}$
$\mathcal{L}$ = lower limit top. (basis $\{[a,b)\}$)

If $f:(\mathbb{R},\mathcal{S}) \to (\mathbb{R},\mathcal{S})$ is continuous, then
$f:(\mathbb{R},\mathcal{L}) \to (\mathbb{R},\mathcal{S})$

<u>Pf</u>: let U be open in $(\mathbb{R},\mathcal{S})$
$f^{-1}(U)$ is open in $(\mathbb{R},\mathcal{S})$
$\therefore f^{-1}(U)$ is open in $(\mathbb{R},\mathcal{L})$. $\boxtimes$

(using fact that everything open in stan. is open in $\mathbb{R}_\ell$)

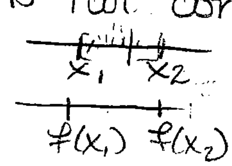
NOTE: The Converse is false:
$g:(\mathbb{R},\mathcal{S}) \to (\mathbb{R},\mathcal{S})$ is not continuous



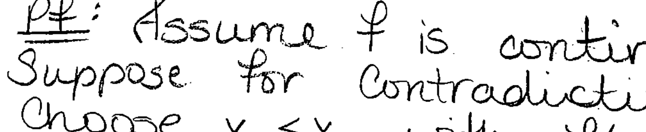
Claim: It is cont. on $g:(\mathbb{R},\mathcal{L}) \to (\mathbb{R},\mathcal{S})$
<u>Pf</u>: let U be open in $(\mathbb{R},\mathcal{S})$
$U = \bigcup (a_\alpha, b_\alpha)$, a union of open intervals.
Each $g^{-1}((a_\alpha, b_\alpha))$ is either an open interval or a half open interval (open on the Right), or it could be empty, so is open in $(\mathbb{R},\mathcal{L})$.
$\therefore g^{-1}(U) = g^{-1}(\bigcup(a_\alpha, b_\alpha)) = \bigcup g^{-1}((a_\alpha, b_\alpha))$ is open in $(\mathbb{R},\mathcal{L})$

(7) If $f(\mathbb{R},\mathcal{S}) \to (\mathbb{R},\mathcal{L})$ is continuous, then f is <u>constant</u>

<u>Pf</u>: Assume f is continuous.
Suppose for Contradiction that f is not constant
Choose $x_1 < x_2$ with $f(x_1) \neq f(x_2)$



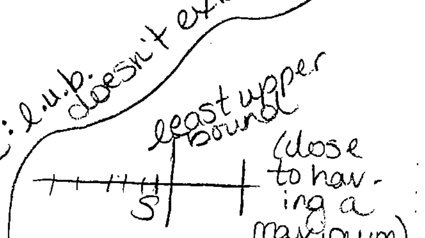
<u>Case I</u>: assume $f(x_1) < f(x_2)$
Let $S = \{x \in [x_1, x_2] \mid f(x) < f(x_2)\}$
$x_1 \in S$, S is nonempty & has an upper bound,
So S has a least upper bound (no gaps),
called m.
<u>case Ia</u>: Suppose $f(m) \geq f(x_2)$ i.e. $m \notin S$
then $m \in f^{-1}([f(x_2), \infty))$, an open set in $(\mathbb{R},\mathcal{S})$

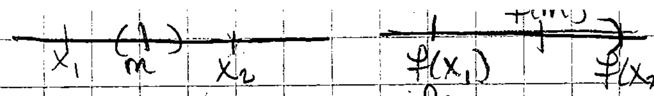


$\therefore \exists\, \varepsilon > 0$, $(m-\varepsilon, m+\varepsilon) \subseteq f^{-1}([f(x_2), \infty))$
So $(m-\varepsilon, m+\varepsilon) \cap S = \emptyset$
and $(m, \infty) \cap S = \emptyset$ So $S \subseteq (-\infty, m-\varepsilon]$
So $m-\varepsilon$ is an upper bound for S $\times$
contradicting m being the least upper bound

$S = \{q \in \mathbb{Q} \mid q < \sqrt{2}\}$ $\therefore$ $\boxtimes$

NOTE: l.u.b. doesn't exist in $\mathbb{Q}$





<u>Case Ib</u>: $f(m) < f(x_2)$

$m \in f^{-1}((-\infty, f(x_2)))$ which is open in $(\mathbb{R}, \mathcal{S})$

So $\exists \varepsilon > 0$ s.t. $(m-\varepsilon, m+\varepsilon) \subseteq f^{-1}((-\infty, f(x_2)))$

$\therefore$ $(m-\varepsilon, m+\varepsilon) \subseteq S$ so $m + \frac{\varepsilon}{2} \in S$ $\times$

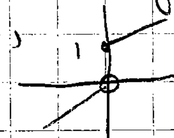
contradicting m being an upper bound for S

<u>Case II</u>: $f(x_1) > f(x_2)$ ... hw...

(8.) $f: (\mathbb{R}, \mathcal{L}) \to (\mathbb{R}, \mathcal{S})$ is continuous if and only if

$\forall x_0 \in \mathbb{R}$, $\lim\limits_{x \downarrow x_0} f(x) = f(x_0)$

$x \downarrow x_0$ — as x approaches x_0 from the right

(NOTE: Satisfies cond. for earlier example, )

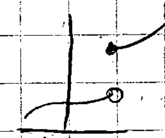
Review of def:

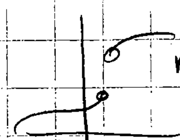
$\lim\limits_{x \downarrow x_0}$ means $\forall \varepsilon > 0, \exists \delta$ s.t. $x_0 \le x < x_0 + \delta \Rightarrow$

$|f(x) - f(x_0)| < \varepsilon$

OR: $x \in [x_0, \delta) \Rightarrow f(x) \in (f(x_0) - \varepsilon, f(x_0) + \varepsilon)$

$\boxed{\text{March, 8}}$:

 okay,

 not okay!

<u>Pf</u>: "$\Rightarrow$" fix x_0 and $\varepsilon > 0$

$x_0 \in f^{-1}((f(x_0) - \varepsilon, f(x_0) + \varepsilon))$ is open in $(\mathbb{R}, \mathcal{L})$

because f is continuous

So $\exists \delta > 0$ s.t. $[x_0, x_0 + \delta) \subseteq f^{-1}((f(x_0) - \varepsilon, f(x_0) + \varepsilon))$

$\therefore$ if $x_0 \in [x_0, x_0 + \delta)$, then $f(x) \in (f(x_0) - \varepsilon, f(x_0) + \varepsilon)$

"$\Leftarrow$" let U be open in $(\mathbb{R}, \mathcal{S})$

let $x_0 \in f^{-1}(U)$, and $f(x_0) \in U$

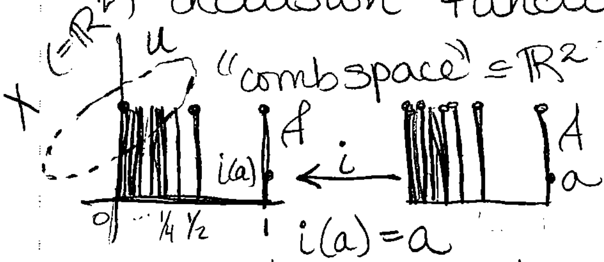
$\exists \varepsilon > 0$, $(f(x_0) - \varepsilon, f(x_0) + \varepsilon) \subseteq U$

$\exists \delta > 0$, if $x_0 \le x < x_0 + \delta$, then $f(x) \in (f(x_0) - \varepsilon, f(x_0) + \varepsilon) \subseteq U$

$\therefore$ $x_0 \in [x_0, x_0 + \delta) \subseteq f^{-1}(U)$

So f is continuous

(9.) If $(X, \mathcal{U})$ is a topological space, and $A \subseteq X$, Then the subspace topology on A is the smallest topology on A that makes the inclusion function continuous



$i : A \to X$, $i(a) = a$

$i(a) = a$

Check the inclusion is continuous:
Let U be open in X
$i^{-1}(U) = A \cap U$. This is open in the subspace topology, $\therefore i$ is cont.

Let $\mathcal{a}$ be a topology on A s.t.
$i : (A, \mathcal{a}) \to (X, \mathcal{U})$ is cont.
Let V be in the Subspace top. on A
$V = A \cap U$ for some open set $U \subseteq X$
$V = i^{-1}(U)$ since i is continuous,
$V = i^{-1}(U) \in \mathcal{a}$ (if not then i wouldn't be cont.)
So $\mathcal{a}$ contains the subspace topology.

(10.) Let $f : X \to Y$ be a function and Let $A \subseteq X$.
The <u>restriction of f to A</u> is the function $f|_A : A \to Y$ defined by $f|_A(a) = f(a)$

Then if f is continuous, then so is $f|_A$.

<u>Pf</u>: $f|_A = f \circ i$, where $i : A \to X$ is the inclusion
i is continuous, so $f \circ i$ is a composition of continuous functions
$\therefore f|_A$ is continuous

(11) A function $\phi : (X, \mathcal{U}) \to (Y, \mathcal{V})$ is a <u>homeomorphism</u> if

(i) ϕ is a bijection

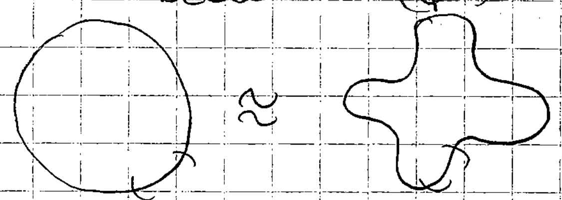
(ii) ϕ is continuous

(iii) ϕ^{-1} is continuous

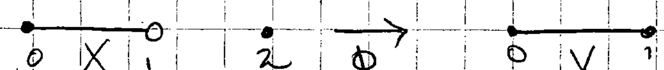
If there exists a homeomorphism from X to Y, then we say X and Y are <u>homeomorphic</u> and we write $\underline{X \approx Y}$

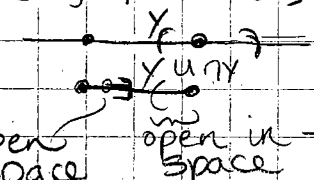
NOTE: if ϕ is a homeomorphism, then ϕ^{-1} is a homeomorphism from Y to X because $(\phi^{-1})^{-1} = \phi$

$\boxed{Ex}$



(12.) A non-example: $X = [0,1) \cup \{2\}$, $Y = [0,1]$





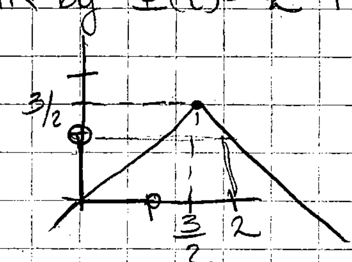
define $\phi : X \to Y$

by $\phi(x) = \begin{cases} x & \text{if } x < 1 \\ 1 & \text{if } x = 2 \end{cases}$

(i) is a bijection.

(ii) ϕ is continuous: define $\Phi : \mathbb{R} \to \mathbb{R}$ by $\Phi(t) = \frac{3}{2} - |\frac{3}{2} - t|$ is continuous by Calculus

ϕ is $\Phi|_X$, ϕ is the restriction of a continuous function, so is continuous.



(iii) ϕ^{-1} is not continuous

$\{2\}$ is open in X, since it is $(\frac{3}{2}, \infty) \cap X$

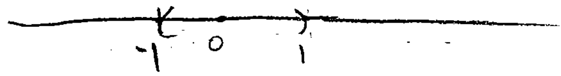
$(\phi^{-1})^{-1}\{2\} = \phi(\{2\}) = \{1\}$ which is not open in Y.

$\therefore$ ϕ is a cont. bijection, but not a homeomorphis

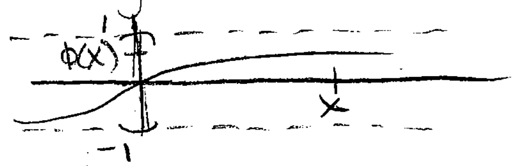
$\boxed{\text{March 10}}$:

(13.) Say $\phi : X \to Y$ is a bijection.
ϕ is a homeomorphism $\Leftrightarrow$ U is open in X
$\Leftrightarrow \phi(U)$ is open in Y
"$\Rightarrow$" Says ϕ^{-1} is continuous
"$\Leftarrow$" Says ϕ is continuous

(14.) $(-1,1) \cong \mathbb{R}$



Note: homeomorphisms can really distort distance
define $\phi : \mathbb{R} \to (-1,1)$ by
$\phi(t) = 2/\pi \tan^{-1}(t)$

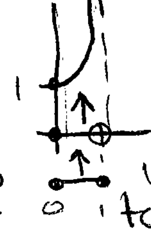


$\phi^{-1}(t) = \tan(\frac{\pi}{2} t)$

(15.) $[0,1) \not\cong [0,1]$

In fact, there is no continuous surjective function from $[0,1]$ to $[0,1)$

Pf: Suppose $\phi : [0,1] \to [0,1)$ were a cont. surjection.
$[0,1] \xrightarrow{\phi} [0,1) \longrightarrow [1,\infty)$
$x \longrightarrow \frac{1}{1-x}$



This would be a continuous unbounded function on the closed interval, contradicting the Extreme Value Thm. (continuous function on a closed interval assumes maximum and min values)

NOTE: There is a (noncontinuous) bijection,
define $\phi : [0,1] \to [0,1)$
by $\phi(1) = 1/2$, $\phi(\frac{1}{2}) = \frac{1}{3}$, $\phi(\frac{1}{3}) = \frac{1}{4}$, etc...

So $\phi(x) = \begin{cases} x & \text{if } x \text{ is not of the for } \frac{1}{n} \\ \frac{1}{n+1} & \text{if } x = \frac{1}{n} \end{cases}$

$\mathbb{R} \not\approx \mathbb{R}^2$

(16.) Is the line homeomorphic to the Plane? (No

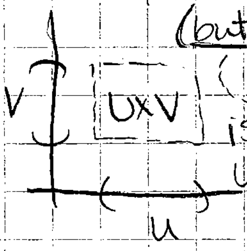
In general, $\mathbb{R}^n \not\approx \mathbb{R}^m$ if $m \neq n$

Products

for sets X and Y, $X \times Y = \{(x,y) \mid x \in X \text{ and } y \in Y\}$

Suppose $(X, \mathcal{U})$ and $(Y, \mathcal{V})$ are topological spaces

How do we put a topology on $X \times Y$?



(but a isn't U×V)

$$\begin{array}{ccc} X \times Y & \xrightarrow{\pi_Y} & Y \\ \downarrow \pi_X & & \\ X & & \end{array}$$

The Projection functions should be continuous

$\pi_X(x,y) = x$

$\pi_Y = (x,y) = y$

So we want a topology that contains all $\pi_X^{-1}(U)$ for U open in X and all $\pi_Y^{-1}(V)$ for V open in Y

- The largest top. on $X \times Y$ is the discrete top. in which these would be open
- We want the smallest, most efficient top. that contains them

General Question:

let X be a set and $\mathcal{S} = \{S_\alpha \mid \alpha \in A\}$ be a collection of subsets of X.

What is the smallest top. on X that contains $\mathcal{S}$

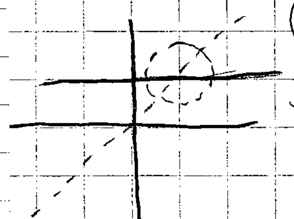
- The topology must contain all intersections of finitely many elements of $\mathcal{S}$

$\{S_1 \cap S_2 \cap \cdots \cap S_k \mid S_i \in \mathcal{S}\}$

$\boxed{\text{Ex}}$ in $\mathbb{R} \times \mathbb{R}$ let $C = \{(r,r) \mid r \in \mathbb{Q}\}$

let $U = \mathbb{R}^2 - C$. The U intersects each $\mathbb{R} \times \{y\}$ and $\{x\} \times \mathbb{R}$ in an open set.

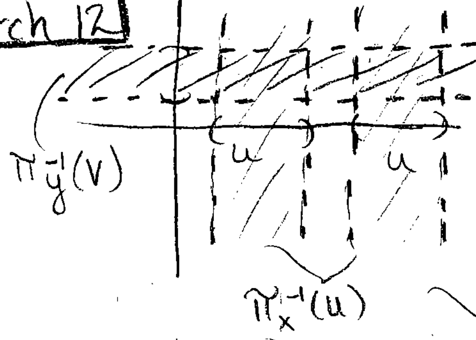
(Either $\mathbb{R}$ or $\mathbb{R}$ - one point)



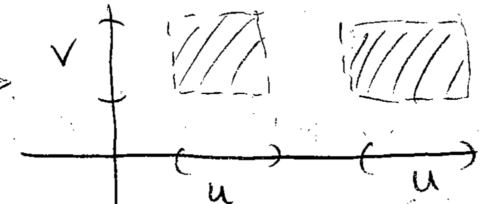
But U is not open in $\mathbb{R}^2$

for example, $(\pi, \pi) \in U$ but every $B((\pi,\pi), \varepsilon)$ meets C.

March 12



Want smallest top in which every $\pi_x^{-1}(U)$, $\pi_y^{-1}(V)$ is open for U open in X, V open in Y



$\pi_x^{-1}(U) \cap \pi_y^{-1}(V) = U \times V$

Back to general question: let $\mathcal{S}$ be a collection of subsets of some set Z. Construct the minimal top., $\mathcal{T}_{\mathcal{S}}$ on Z that contains $\mathcal{S}$.

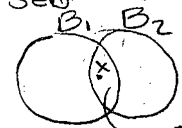
- $\mathcal{T}_{\mathcal{S}}$ must contain every $S_1 \cap \cdots \cap S_n$ where $S_i \in \mathcal{S}$
- $\mathcal{T}_{\mathcal{S}}$ must contain every unions of these

Do these $S_1 \cap \cdots \cap S_n$, by any chance, form a basis?
<u>Yes</u>, (*provided their union is $\mathcal{S}$, $\bigcup_{S \in \mathcal{S}} S = X$)

let $\mathcal{B} = \{S_1 \cap \cdots \cap S_n \mid S_i \in \mathcal{S}\}$

let $S_1 \cap \cdots \cap S_n$, $T_1 \cap \cdots \cap T_n \in \mathcal{B}$

$(S_1 \cap \cdots \cap S_n \cap T_1 \cap \cdots \cap T_n) \in \mathcal{B}$



$x \in B_1 \cap B_2 \subseteq B \cap B$

- <u>def</u>: A collection $\mathcal{S}$ of subsets of a set Z is a sub-basis if $\bigcup_{S \in \mathcal{S}} S = Z$ (which can alway be assured by replacing $\mathcal{S}$ with $\mathcal{S} \cup \{X\}$)

- When $\mathcal{S}$ is a sub-basis, the collection $\mathcal{B} = \{S_1 \cap \cdots S_n \mid S_i \in \mathcal{S}\}$ is a basis.

- The topology generated <u>by $\mathcal{S}$</u> is defined to be the topology generated by this basis.

- Any topology that contains $\mathcal{S}$ must contain $\mathcal{B}$ and hence must contain the topology generated by $\mathcal{B}$.

Applied to our product situation, let

$\mathcal{S} = \{\pi_X^{-1}(U) \mid U \text{ is open in } X\} \cup \{\pi_Y^{-1}(V) \mid V \text{ open in } Y\} \subseteq$ subsets of $X \times Y$.

$\mathcal{S}$ contains $X \times Y = \pi_X^{-1}(X)$ so $\mathcal{S}$ is a sub-basis.

Notice: if $U_1, \dots, U_n$ are open in X,

$$\pi_X^{-1}(U_1) \cap \pi_X^{-1}(U_2) \cap \dots \cap \pi_X^{-1}(U_n) = \pi_X^{-1}(\overbrace{U_1 \cap \dots \cap U_n}^{\text{open set in } X})$$

$$= \pi_X^{-1}(U) \text{ for an open } U \text{ in } X$$

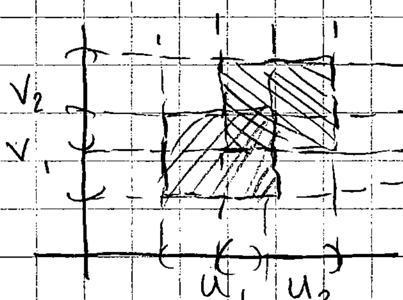
The basis we get from $\mathcal{S}$:

$$\mathcal{B} = \{\pi_X^{-1}(U_1) \cap \dots \cap \pi_X^{-1}(U_n) \cap \pi_Y^{-1}(V_1) \cap \dots \cap \pi_Y^{-1}(V_n) \mid U_i \text{ open in } X, V_j \text{ open in } Y\}$$

$$= \{\pi_X^{-1}(U) \times \pi_Y^{-1}(V) \mid U \text{ open in } X, V \text{ open in } Y\}$$

$$= \{U \times V \mid U \text{ open in } X, V \text{ open in } Y\}$$

def.: The product topology on $X \times Y$ is the topology generated by the basis $\{U \times V \mid U \text{ open in } X, V \text{ open in } Y\}$

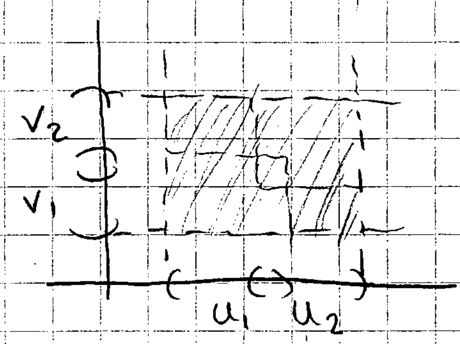


$(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2)$

BUT: $(U_1 \times V_1) \cup (U_2 \times V_2) \subseteq (U_1 \cup U_2) \times (V_1 \cup V_2)$

but not equal

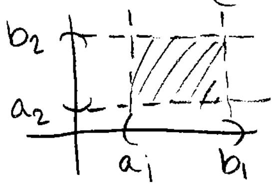
contains extra stuff as well



$\boxed{Ex}$ ① $(\mathbb{R}^m, \text{std}) \times (\mathbb{R}^n, \text{std}) = (\mathbb{R}^{m+n}, \text{std})$

pf: Use the fact (HW) that if $\mathcal{B}_X$ is a basis for X and $\mathcal{B}_Y$ is a basis for Y, then $\{B_1 \times B_2 \mid B_1 \in \mathcal{B}_X, B_2 \in \mathcal{B}_Y\}$ is a basis for the product topology.

A basis for $\underbrace{(\mathbb{R}^m, \text{std})}_{}$ $= \mathcal{B}_m$ is all $\{(a,b) \times \dots \times (a_m, b_m) \mid a_i < b_i\}$



(Can Prove w/ Basis Rec. Thm.)

$\{B_1 \times B_2 \mid B_1 \in \mathcal{B}_m, B_2 \in \mathcal{B}_n\}$ is a basis for the product topology

$= \{(a_1, b_1) \times \dots \times (a_m, b_m) \times (a_{m+1}, b_{m+1}) \times \dots \times (a_{m+n}, b_{m+n}) \mid a_i < b_i\}$

is a basis for the standard top. on $\mathbb{R}^{m+n}$