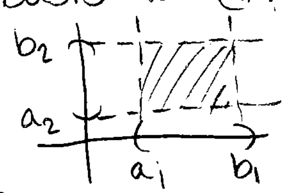
$\boxed{Ex}$ ① $(\mathbb{R}^m, std) \times (\mathbb{R}^n, std) = (\mathbb{R}^{m+n}, std)$

pf: Use the fact (HW) that if $\mathcal{B}_X$ is a basis for X and $\mathcal{B}_Y$ is a basis for Y, then $\{B_1 \times B_2 \mid B_1 \in \mathcal{B}_X, B_2 \in \mathcal{B}_Y\}$ is a basis for the product topology.

A basis for $(\mathbb{R}^m, std)$ is all $\underbrace{\{(a,b) \times \ldots \times (a_m, b_m) \mid a_i < b_i\}}_{\mathcal{B}_m}$



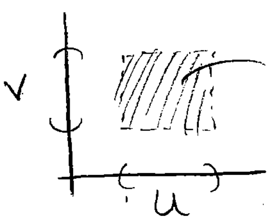
(Can Prove w/ Basis Rec. Thm.)

$\{B_1 \times B_2 \mid B_1 \in \mathcal{B}_m, B_2 \in \mathcal{B}_n\}$ is a basis for the product topology

$= \{(a_1,b_1) \times \ldots \times (a_m,b_m) \times (a_{m+1}, b_{m+1}) \times \cdots \times (a_{m+n}, b_{m+n}) \mid a_i < b_i\}$
is a basis for the standard top. on $\mathbb{R}^{m+n}$

$\boxed{\text{March 22}}$:

$\mathcal{B} = \{U \times V \mid U \text{ open in } X \text{ and } V \text{ open in } Y\}$ is a basis for the product topology $X \times Y$



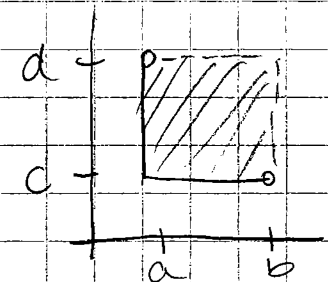
$\boxed{Ex}$: ① $\mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^{m+n}$ (std $\times$ std = std)
② $X \times Y$ is discrete $\Leftrightarrow$ X & Y are discrete

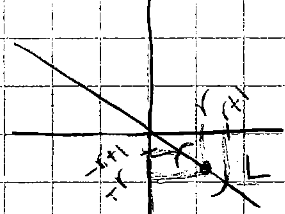
"$\Leftarrow$" for every $(x,y) \in X \times Y$, (non-empty X and Y)
$\{(x,y)\} = \{x\} \times \{y\}$. Since X and Y are discrete, $\{x\}$ and $\{y\}$ are open. $\{(x,y)\}$ is open

"$\Rightarrow$" $X \times Y$ is discrete. Let $x \in X$
$\{x\} \times Y$ is an open subset of $X \times Y$ (every subset open) choose $y_0 \in Y$ (assuming $Y \neq \emptyset$).
There exists a basic open set $U \times V$ with $(x, y_0) \in U \times V \subseteq \{x\} \times Y$.
$\therefore x \in U \subseteq \{x\}$, so $U = \{x\}$ is open in X.
Similarly, Y is discrete

[Ex] $\mathbb{N} \times \mathbb{N}$ is discrete and countable,
so $\mathbb{N} \times \mathbb{N} \cong \mathbb{N}$
(both are discrete, so any bijection is a homeomorphism)

③ Sorgenfrey plane $(\mathbb{R}, \mathcal{L}) \times (\mathbb{R}, \mathcal{L}) = X$
A basis for $\mathcal{L}$ is $\{[a,b) \mid a<b\}$.
A basis for $(\mathbb{R}, \mathcal{L}) \times (\mathbb{R}, \mathcal{L})$ is
$\{[a,b) \times [c,d) \mid a,b,c,d \in \mathbb{R}\}$
let $L = \{(r,-r) \mid r \in \mathbb{R}\} \subseteq X$





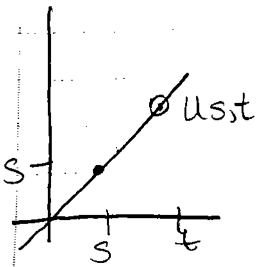
The Subspace Topology on L is the discrete topology. What sort of basic open sets can we get?
closed intervals
open intervals
half open intervals
☆ single points

for each r, $\{(r,-r)\} = L \cap ([r, r+1) \times [-r, -r+1))$
So each point of $\mathcal{L}$ is an open set.

<u>remark</u>: any line of negative slope has discrete topology

positive slope line: no corner trick, only get half-open intervals

Let $L_2 = \{(r,r) \mid r \in \mathbb{R}\}$. Will show $\mathcal{L}_2$ w/ subspace topology is homeomorphic to $(\mathbb{R}, \mathcal{L})$



For $s, t \in \mathbb{R}$, define

$U_{s,t} = \{(x,x) \mid s \leq x < t\}$

exercise: $\mathcal{L}_2 \cap [a,b) \times [c,d) = U_{\max\{a,c\}, \min\{b,d\}}$

So, $\{U_{s,t} \mid s,t \in \mathbb{R}\}$ is a basis for the subsp. top

define $h: \mathcal{L}_2 \to (\mathbb{R}, \mathcal{L})$ by $h((r,r)) = r$

$h^{-1}([a,b))$ is open $(h^{-1}([a,b)) = U_{a,b})$ (h cont.)

$h(U_{a,b}) = [a,b)$ is open (h^{-1} cont.)

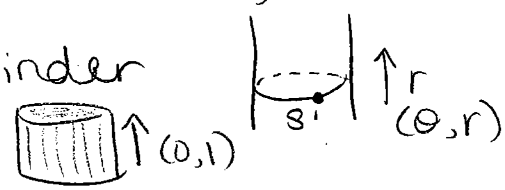
$\therefore$ h is a homeomorphism

$\therefore$ $\mathcal{L}_2 \cong (\mathbb{R}, \mathcal{L})$

Similarly, each vertical, horizontal, or positive sloping line is homeomorphic to $(\mathbb{R}, \mathcal{L})$

4. $S^1 \times \mathbb{R}$ is the infinite cylinder

$\cong S^1 \times (0,1)$ open cylinder



$S^1 \times S^1$ is the <u>torus</u>

- fixed θ_2 coordinate specifies a circle here
- fixed θ_1 coordinate specifies a circle here

<u>Surfaces</u>: (orientable)

Sphere S^2 torus   genus 2

5. $\mathbb{Q} \times \mathbb{Q} \cong \mathbb{Q}$

for $J = \mathbb{R}/\mathbb{Q}$ (uncountable)

$J \times J \cong J$ (not easy facts)

6. C = Cantor Set

$C \times C \cong C$

March 24: $f: Z \to X \times Y$ (interested in when continuous)

$f: Z \to \mathbb{R} \times \mathbb{R}$
$f(z) = f_1(z), f_2(z)$
$\quad\quad\quad\ \ \in X \quad \in Y$

let $\pi_x : X \times Y \to X$ $\quad \pi_y : X \times Y \to Y$
$\quad\quad (x,y) \longmapsto x \quad\quad (x,y) \longmapsto y$
be the projections

could write: $(x,y) = (\pi_x(x,y), \pi_y(x,y))$
$f(z) = (\pi_x(f(z)), \pi_y(f(z)))$
$= ((\pi_x \circ f)(z), (\pi_y \circ f)(z))$

- So, if $f: Z \to X \times Y$ is a function, we define the x-coordinate function of f to be $\pi_x \circ f : Z \to X$ and the y-coordinate function of f to be $\pi_y \circ f : Z \to Y$

- On the other hand, if we start w/ any 2 functions $g: Z \to X$ and $h: Z \to Y$, we can define an f w/ g and h as coordinate functions by
  $f(z) = (g(z), h(z))$

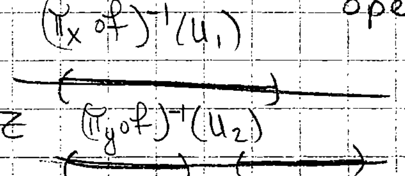
Mapping Into Products Thm.:
$f: Z \xrightarrow{} X \times Y$ is continuous $\Leftrightarrow$ the coordinate functions of f are continuous

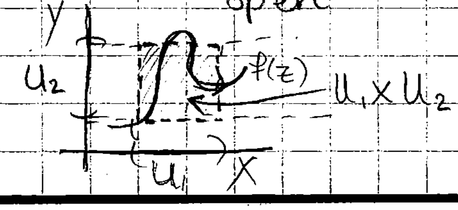
Pf: "$\Rightarrow$" Assume f is cont., then π_x and π_y are continuous (b/c $\pi_x^{-1}(U) = U \times Y$ open for all open sets U in X) so $\pi_x \circ f$ and $\pi_y \circ f$ are continuous

"$\Leftarrow$" Assume that $\pi_x \circ f$ and $\pi_y \circ f$ are continuous
It is sufficient to show that $f^{-1}(U_1 \times U_2)$ is open for each basic open set in $X \times Y$.
Suppose U_1 open in X, U_2 open in Y.

$z \in f^{-1}(U_1 \times U_2) \Leftrightarrow f(z) \in U_1 \times U_2$
$\Leftrightarrow ((\pi_x \circ f)(z), (\pi_y \circ f)(z)) \in U_1 \times U_2$
$\Leftrightarrow (\pi_x \circ f)(z) \in U_1$ and $(\pi_y \circ f)(z) \in U_2$
$\Leftrightarrow z \in (\pi_x \circ f)^{-1}(U_1)$ and $z \in (\pi_y \circ f)^{-1}(U_2)$
$\Leftrightarrow z \in ((\pi_x \circ f)^{-1}(U_1)) \cap ((\pi_y \circ f)^{-1}(U_2))$

$\therefore f^{-1}(U_1 \times U_2) = \underbrace{(\pi_x \circ f)^{-1}(U_1)}_{\text{open}} \cap \underbrace{(\pi_y \circ f)^{-1}(U_2)}_{\text{open}}$ is open ■

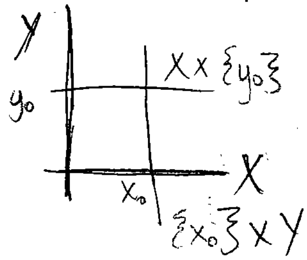




Remark 1:

The product topology on $X \times Y$ is the only topology that makes the mapping into products thm. true.

Remark 2:

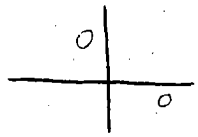


Ex $f: \mathbb{R}^2 \to \mathbb{R}$ defined by $f: \begin{cases} \frac{xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

for each x_0, $f|_{\{x_0\} \times \mathbb{R}}$ is continuous:

$f(x_0, y) = \frac{x_0 y}{x_0^2 + y^2}$ is cont. when $x_0 \neq 0$

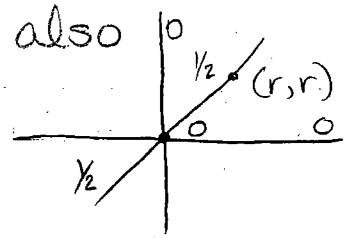
$f(0,y) = \begin{cases} 0 \text{ when } y \neq 0 \\ 0 \text{ when } y = 0 \end{cases}$



Similarly, $f|_{\mathbb{R} \times \{y_0\}}$ is continuous also

But f is not continuous bc

$f(r,r) = \begin{cases} \frac{r^2}{r^2+r^2} = \frac{1}{2} & \text{when } r \neq 0 \\ 0 & \text{when } r = 0 \end{cases}$



Remark 3: On an infinite product $\prod X_\alpha$, the product topology is not the one with basis $\{\prod U_\alpha \mid U_\alpha \text{ open in } X\}$

$U_\alpha \subseteq X_\alpha$ Each point in the product, $\prod X_\alpha$

$\prod U_\alpha \subseteq \prod X_\alpha$ has one coordinate in each factor X_α.

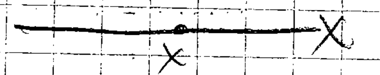
For Subsets $U_\alpha \subseteq X_\alpha$, $\prod U_\alpha$ is the subset of $\prod X_\alpha$ (for which each coordinate lies in the subset U_α of X_α). Consisting of the points whose X_α-coordinate lies in U_α for every α.

- That basis produces the <u>box topology</u>, not the product topology

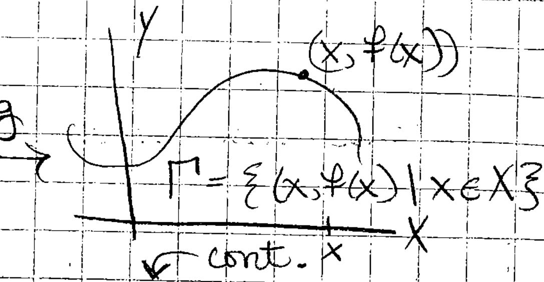
Example:

March 29: $f: X \to Y$

define



$\xrightarrow{g}$



$g: X \to X \times Y$
by $g(x) = (x, f(x))$

coordinate functions of g are id_X and f, so if f is continuous, then g is cont.

- In fact, if f is cont., then g is a homeomorphism onto the graph of f. Its inverse is $\pi_X|_\Gamma$ which is continuous.

<u>Example</u>: let $f: \mathbb{R} \to \mathbb{R} \times \mathbb{R} \times \cdots = \{(r_1, r_2, \dots) \mid r_i \in \mathbb{R}\}$

let $f(t) = (t, t, \dots)$

ex: $(\frac{1}{2}, \frac{1}{4}, -\frac{1}{6}, \frac{1}{5}, \dots)$

- Notice that $f^{-1}((-1,1) \times (-\frac{1}{2}, \frac{1}{2}) \times (-\frac{1}{3}, \frac{1}{3}) \times \cdots)$
  $= \bigcap_{n=1}^{\infty} (-\frac{1}{n}, \frac{1}{n}) = \{0\}$ is not open in $\mathbb{R}$

So if $(-1,1) \times (-\frac{1}{2}, \frac{1}{2}) \times \cdots$ were an open set, this f would not be continuous, even though all coordinate functions of f are the identity map of $\mathbb{R}$!

In fact, for any product $\prod_{\alpha \in a} X_\alpha$, the product topology is defined to be the topology generated by the sub-bases $\left\{ \pi_\beta^{-1}(U) \,\middle|\, \beta \in a \text{ and } U \text{ is open in } X_\beta \right\}$

<u>Closed Sets and limit points</u>

<u>def.</u>: a subset $A \subseteq X$ is <u>closed</u> if $X - A$ is open

$\boxed{Ex}$ $(\;]$ neither open nor closed in $\mathbb{R}$

$[\;)$ is open and closed in $(\mathbb{R}, \mathcal{L})$

Remember: A set can be open (or closed) in a subspace w/out being open (or closed) in the Ambient space

$\boxed{Ex}$ $A = (0,1) \cup \{2\} \subseteq \mathbb{R}$

- $(0,1)$ open in A and in $\mathbb{R}$
- $\{2\}$ is open in A, but not $\mathbb{R}$
  $A \cap (3/2, 5/2)$

- So $\{2\}$ is closed in A ($A - \{2\} = (0,1)$ open) and closed in $\mathbb{R}$ ($\mathbb{R} - \{2\} = (-\infty, 2) \cup (2, \infty)$)

- $(0,1)$ is closed in A ($A-(0,1)=\{2\}$ is open in A) but not closed in $\mathbb{R}$ ($\mathbb{R}-(0,1)=(-\infty,0]\cup[1,\infty)$ is not open)

- In any X, X and $\emptyset$ are closed subset because $X-X=\emptyset$ is open
  $X-\emptyset=X$ is open
- If X has the discrete topology, every subset is closed (eg. $\mathbb{N}$)

$\boxed{Ex}$: $0 \; [\,[\,[\quad]\quad]$ with $\frac{1}{4}, \frac{1}{3}, \frac{1}{2}, 1$ $\bigcup_{n=1}^{\infty}\left[\frac{1}{n+1},\frac{1}{n}\right]=(0,1]$ is not closed in $\mathbb{R}$

- Unions of finitely many closed sets are closed
- Intersection of any collection of closed sets is closed

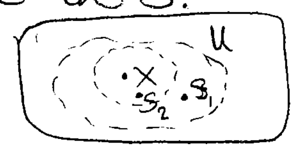
<u>Pf</u>: let $C_1,\dots,C_n$ be closed sets in X.
$X-\bigcup_{i=1}^{n}C_i=\bigcap_{i=1}^{n}\underbrace{(X-C_i)}_{\text{open}}$ So is intersection of finitely many open sets, so is open
$\therefore \bigcup_{i=1}^{n}C_i$ is closed
(same w/ $\bigcap_{i=1}^{n}C_i\dots$)

$\boxed{\text{March 31}}$:

<u>def</u>: let $x\in X$. A <u>neighborhood</u> of x is an open set that contains x.

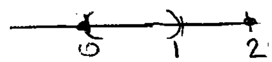
☆ <u>def</u>: let $S\subseteq X$ and let $x\in X$. We say x is a <u>limit point of S</u> if every neighborhood of x contains a point of S, other than x itself if x happens to be in S.

$\forall$ open U, $x\in U \Rightarrow (U-\{x\})\cap S$



- can take smaller & smaller neighborhoods with an $s\in S$ in each neighborhood

$\boxed{Ex.}$: $S=(0,1)\cup\{2\}\subseteq\mathbb{R}$



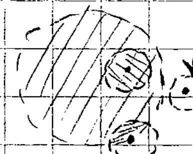
$[0,1]$ are lim. pt.
- 0 is a limit point
- 2/3 & everything else in $(0,1)$ is a limit point
- 1 is a limit point
- BUT 2 is <u>not</u> a limit point

def: the set of limit points of S is called the limit set of S (or derived set) and is denoted by S'

★ def: The set $S \cup S'$ is called the closure of S and is denoted by $\overline{S}$ or $cl_X(S)$

Ex: $cl_{\mathbb{R}}((0,1)) = [0,1]$
$cl_{[0,1)}((0,1)) = [0,1)$

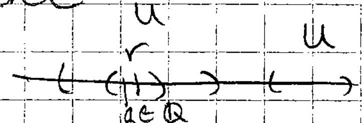
$\boxed{Ex}^2$ $S = \{x \in \mathbb{R}^2 \mid \|x\| < 1\} \subseteq \mathbb{R}^2$
$\overline{S} = \{x \in \mathbb{R}^2 \mid \|x\| \leq 1\}$



i.e. "add the boundary to S"
boundary of $S = \overline{S} \cap \overline{X-S}$

NOTE: boundary points are limit points of S and its complement

$\boxed{Ex}^3$ $S = \mathbb{Q} \subseteq \mathbb{R}$ $S' = \mathbb{R}$

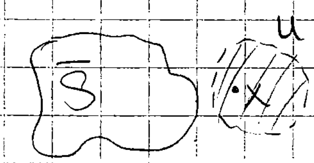


$\overline{\mathbb{Q}} = \mathbb{R}$
Similarly, $\mathbb{R}-\mathbb{Q} = S$, then $\overline{\mathbb{R}-\mathbb{Q}} = \mathbb{R}$

closure is the entire line

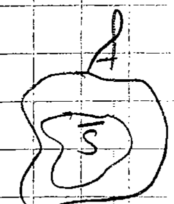
Propositions:

① $\overline{S}$ is closed



pf: Suppose $x \notin \overline{S}$, then $x \notin S$
also, $x \notin S'$, so $\exists$ a neighborhood, U of x s.t. $U - \{x\}$ contains no points of S
so since $x \notin S$, U contains no points of S
$\therefore X - \overline{S}$ is open [$\forall x \in X-\overline{S}$, choose an open set U_x w/ $x \in U_x \subseteq X-\overline{S}$ and $X - \overline{S} = \bigcup_{x \in X-\overline{S}} U_x$]

② If A is a closed subset of X and $S \subseteq A$, then $\overline{S} \subseteq A$
(i.e. $\overline{S}$ is the smallest closed set containing S)



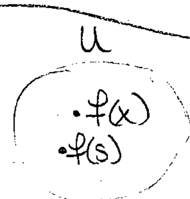
pf: We will show $X - A \subseteq X - \overline{S}$, which means $\overline{S} \subseteq A$
Suppose $x \notin A$. Then $x \in \underbrace{X-A}_{\text{open}} \subseteq X - S$.
So $x \notin S$, and
$X-A$ is a nbd. of x containing no points of S, so $x \notin S'$
$\therefore x \notin S$ and $x \notin S'$ so $x \notin S \cup S' = \overline{S}$. ∎

Remark: $\bar{S} = \bigcap_{\substack{A \text{ closed} \\ \text{and } S \subseteq A}} A$

Compactness: the general def. encompasses various equivalent definitions for special cases used in analysis
(e.g. closed & bounded ← may not be eq. to compactness in more general situations
every seq. has a converging subsequence)

Calc. I: Mean Value Thm. is what everything depends on:
If $f: [a,b] \to \mathbb{R}$ is continuous & differentiable on (a,b), then $\exists c \in [a,b]$ s.t. $\underbrace{f(b) - f(a)}_{\text{total ("global") change of } f} = \underbrace{f'(c)}_{\text{local rate of change}}(b-a)$
(a local to global Thm.)
← why fundamentally important

(44.) Homework:
$f: X \to Y$ cont. $f(\bar{S}) \subseteq \overline{f(S)}$



- $f(S) \subseteq \overline{f(S)}$ want to show $f(S') \subseteq \overline{f(S)}$
let $x \in S'$. Argue $f(x) \in \overline{f(S)}$

$f: X \to Y$ of $f^{-1}(C)$ is closed for all C closed in Y

- $f^{-1}(C) = f^{-1}(Y - (Y - C)) = X - f^{-1}(Y - C)$

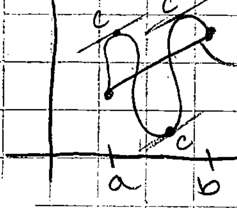
$\bar{S}$ is one of the A's

(44.) c $\bar{S} = \bigcap_{\substack{A \subseteq X \\ S \subseteq A \\ A \text{ closed}}} A$ 1.) show $\bar{S} \subseteq A$ for all such A so $\bar{S} \subseteq \bigcap A$

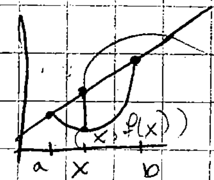
closed dis...

April 2 ...continuation of mean value Thm...

MVT: $f:[a,b]\to\mathbb{R}$ cont. & differentiable
Then $\exists\, c \in [a,b]$ s.t. $f'(c) = \dfrac{f(b)-f(a)}{b-a}$



$m = \dfrac{f(b)-f(a)}{b-a}$



$\left(x, f(a) + \dfrac{f(b)-f(a)}{b-a}(x-a)\right)$

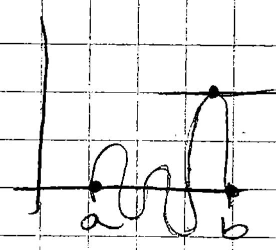
Pf: 1.) Reduce to a special case: $f(a)=f(b)=0$
(Show if MVT is true when $f(a)=f(b)=0$, then it is true for all f)

define $g(x) = f(x) - \left(f(a) + \dfrac{f(b)-f(a)}{b-a}(x-a)\right)$
and $g(a)=g(b)=0$

Assuming g satisfies the MVT, $\exists\, c \in (a,b)$
s.t. $0 = \dfrac{g(b)-g(a)}{b-a} = g'(c) = f'(c) - \left(0 + \dfrac{f(b)-f(a)}{b-a}\cdot 1\right)$

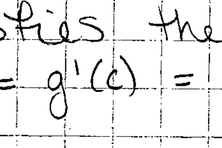
$\therefore f'(c) = \dfrac{f(b)-f(a)}{b-a}$

2.) Verify the MVT when $f(a)=f(b)=0$
if $f(x)=0$ for $a \le x \le b$
The $f'(x)=0$ for all x, so c can be any number in (a,b)



Otherwise, choose a c in (a,b) with $|f(c)|$ maximal (using the extreme value Thm.: A cont. function on a closed interval assumes maximum and minimum values)

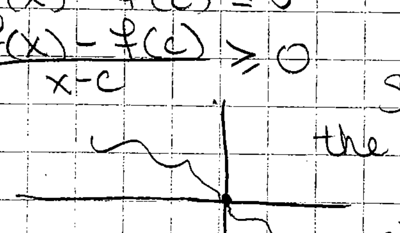
case I: $f(c) > 0$



for $x<c$, $f(x) \le f(c)$
$f(x) - f(c) \le 0$
$\dfrac{f(x)-f(c)}{x-c} \geq 0$

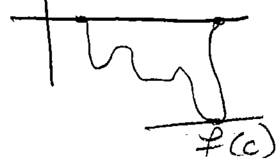
for $x>c$, $\dfrac{f(x)-f(c)}{x-c} \le 0$

$\dfrac{f(x)-f(c)}{x-c}$



Since $f'(c)$ exists, $f'(c)$ is the $\lim\limits_{x\to c} \dfrac{f(x)-f(c)}{x-c} = 0$
since lim of both nonneg & nonpos.

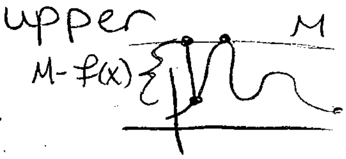
case II: $f(c) < 0$ (similar)



- We used

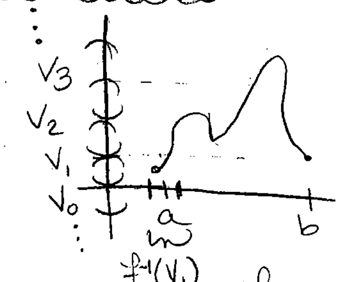
<u>Thm.</u> let $f: [a,b] \to \mathbb{R}$ be continuous. Then $\exists c$ in $[a,b]$ s.t. $f(c) \geq f(x)$ $\forall\, a \leq x \leq b$

<u>Pf:</u> <u>Every cont. function on a closed interval must be bounded</u>. let M be the least upper bound for the values.



Suppose there is no x in $[a,b]$ s.t. $f(x) = M$ (for contradiction),
define $g: [a,b] \to \mathbb{R}$ by $g(x) = \frac{1}{M - f(x)}$
$g(x)$ would be continuous and <u>unbounded</u> function on $[a,b]$ (g goes to ∞ when close to M) a contradiction.

<u>Thm.</u>: Every continuous function on a closed interval, $[a,b]$ must be bounded



- <u>Pf</u>: for $n \in \mathbb{Z}$, define $V_n = (n-1, n+1)$
define $U_n = f^{-1}(V_n)$
Since f is <u>continuous</u>, U_n's are open subsets of $[a,b]$.
Also, every x is in some U_n, Since $f(x) \in (n-1, n+1)$ for some n

So $[a,b] = \bigcup_{n \in \mathbb{Z}} U_n$

<u>def.</u>: $\{U_n\}_{n \in \mathbb{Z}}$ is a collection of open subsets of $[a,b]$ whose union is $[a,b]$. This is called an <u>open cover</u> of $[a,b]$.

<u>def</u>: Suppose we know that some finite subcollection is an open cover. (the collection has a <u>finite subcover</u>).

So $[a,b] = U_{n_1} \cup U_{n_2} \cup \cdots \cup U_{n_k}$

- Then every $f(x)$ lies in atleast one of (n_1-1, n_1+1), (n_2-1, n_2+1), ..., (n_k+1, n_k-1)
So all $f(x)$ are $\leq \max n_i + 1$ and $\geq \min n_i - 1$