Math 4853 homework

62. Prove that every uncountable subset of $\mathbb{R}$ has a limit point in $\mathbb{R}$. (Let A be an uncountable subset of $\mathbb{R}$, and for $n \in \mathbb{Z}$ put $A_n = A \cap [n, n+1]$.)

63. Let $\{ x_n \}$ be a sequence in a metric space X. Prove that if $x_n \to x$, then $\{ x_n \}$ is Cauchy.

64. Give $\mathbb{R}^k$ the metric $d(x,y) = \| x - y \|$. Let $\{ z_n \}$ be a sequence of points in $\mathbb{R}^k$, written in coordinates as $z_n = (z_n^1, z_n^2, \ldots, z_n^k)$. Prove that $\{ z_n \}$ is Cauchy if and only if each $\{ z_n^i \}$ is a Cauchy sequence in $(\mathbb{R}, |x - y|)$.

65. Let $\{f_n\}$ be a sequence of functions in $C([0,1], \mathbb{R}^k)$ (the set of continuous functions from $[0,1]$ to $\mathbb{R}^k$. Prove that if $\{f_n\} \to f$ uniformly, then $\{f_n\} \to f$ pointwise.

66. Let $f_n \colon [0,1] \to \mathbb{R}$ be $f_n(x) = x^n$, and let $f \colon [0,1] \to \mathbb{R}$ be defined by $f(x) = 0$ if $x < 1$ and $f(1) = 1$. Using the definitions, prove that $f_n \to f$ pointwise but not uniformly.