Math 4853 homework solutions (version of February 12, 2010)

1. Let $f\colon \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = x^2$. Prove (directly from the definition) that f is continuous.

Fix x_0, and suppose for now that $x_0 \neq 0$. Note first that if $|x - x_0| < |x_0|$, then $|x| = |x - x_0 + x_0| \leq |x - x_0| + |x_0| < |x_0| + |x_0| = 2|x_0|$. Now, given $\epsilon > 0$, put $\delta = \min\{|x_0|, \epsilon/(3|x_0|)\}$; since $x_0 \neq 0$, $\delta > 0$. Then, if $|x - x_0| < \delta$, we have

$$|x^2 - x_0^2| = |x + x_0|\ |x - x_0| < 3|x_0|\ \epsilon/(3|x_0|) = \epsilon\ .$$

Suppose now that $x_0 = 0$. Given $\epsilon > 0$, put $\delta = \sqrt{\epsilon}$. If $|x - 0| < \delta$, then $|x^2 - 0^2| = x^2 < (\sqrt{\epsilon})^2 = \epsilon$.

2. Let $f\colon \mathbb{R} \to \mathbb{R}$ be the function defined as follows.

$$f(x) = \begin{cases} (q-1)/q & x \text{ is rational and } x = \pm\dfrac{p}{q} \text{ in lowest terms with } p \geq 0 \text{ and } q > 0 \\ 1 & x \text{ is irrational} \end{cases}$$

Prove (directly from the definition) that if x is irrational, then f is continuous at x.

Given ϵ, choose a positive integer N with $1/N < \epsilon$. Let S be the set of rational numbers in the interval $(x - 1, x + 1)$ with the following property:

> If r_i is written as p_i/q_i with p_i and q_i integers in lowest terms, with $q_i > 0$, then $q_i \leq N$.

We note that S is nonempty, since there is at least one integer in the interval $(x - 1, x + 1)$ for which the denominator is 1, and S is finite, since the interval $(x - 1, x + 1)$ has finite length. So we can write $S = \{r_1, \ldots, r_k\}$.

Each $|r_i - x| > 0$, since x is irrational, so the number $S_{min} = \min\{|r_i - x|\}_{r_i \in S}$ is positive. Put $\delta = \min\{S_{min}, 1\}$.

Case I: z is irrational

In this case $|f(z) - f(x)| = |1 - 1| = 0 < \epsilon$.

Case II: z is rational

Since $|z - x| < \delta \leq 1$, $z \in (x - \delta, x + \delta)$, but since $|z - x| < \delta$, z cannot equal any r_i. So $z = p/q$ in lowest terms with $q > N$. Therefore $|f(z) - f(x)| = |(q-1)/q - 1| = 1/q < 1/N < \epsilon$.

3. Let $f\colon \mathbb{R} \to \mathbb{R}$ be the function defined as follows.

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

[where $\mathbb{Q}$ denotes the set of rational numbers] Use proof by contradiction to prove that f is not continuous at any x_0.

Fix x_0 and suppose for contradiction that f is continuous at x_0. Then there exists $\delta > 0$ such that if $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < 1$.

Case I: x_0 is rational.

Choose an irrational x in the interval $(x_0 - \delta, x_0 + \delta)$. Then $|x - x_0| < \delta$, but $|f(x) - f(x_0)| = |0 - 1| = 1$, a contradiction.

Case II: x_0 is irrational.

Choose a rational x in the interval $(x_0 - \delta, x_0 + \delta)$. Then $|x - x_0| < \delta$, but $|f(x) - f(x_0)| = |1 - 0| = 1$, a contradiction.

4. Let $f\colon \mathbb{R} \to \mathbb{R}$ be the function defined as follows.

$$f(x) = \begin{cases} 1/q & x \text{ is rational and } x = p/q \text{ in lowest terms with } q > 0 \\ 0 & x \text{ is irrational} \end{cases}$$

Use proof by contradiction to prove that if x is rational, then f is not continuous at x.

Fix a rational number x_0 and suppose for contradiction that f is continuous at x_0. Write $x_0 = p/q$ in lowest terms with $q > 0$. Then there exists $\delta > 0$ such that if $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < 1/q$. Choose an irrational x in the interval $(x_0 - \delta, x_0 + \delta)$. Then $|x - x_0| < \delta$, but $|f(x) - f(x_0)| = |0 - 1/q| = 1/q$, a contradiction.