Math 4853 homework solutions (version of April 5, 2010)

18. Verify that if U is open in the standard topology on $\mathbb{R}$, then it is open in the lower limit topology on $\mathbb{R}$.

Let U be open in the standard topology, and let $x \in U$. Then there exists $\epsilon > 0$ such that $(x-\epsilon, x+\epsilon) = B(x,\epsilon) \subseteq U$. Therefore $x \in [x, x+\epsilon) \subseteq U$, so U is open in the lower-limit topology.

19. Verify that the cofinite topology on $\mathbb{R}$ is a topology. You will probably want to make use of DeMorgan's identities: If $\{A_\alpha\}_{\alpha\in\mathcal{A}}$ are subsets of a set X, then $X - \cup_{\alpha\in\mathcal{A}} A_\alpha = \cap_{\alpha\in\mathcal{A}}(X - A_\alpha)$ and $X - \cap_{\alpha\in\mathcal{A}} A_\alpha = \cup_{\alpha\in\mathcal{A}}(X - A_\alpha)$.

Let $\mathcal{U}$ be the cofinite topology on $\mathbb{R}$. By definition, $\emptyset \in \mathcal{U}$. Since $\mathbb{R} - \mathbb{R} = \emptyset$ is finite, $\mathbb{R} \in \mathcal{U}$.

Suppose that $\{U_\alpha\}_{\alpha\in\mathcal{A}} \subseteq \mathcal{U}$. If all $U_\alpha = \emptyset$,then $\cup_{\alpha\in\mathcal{A}} U_\alpha = \emptyset \in \mathcal{U}$. Otherwise, $U_\beta \neq \emptyset$ for some $\beta \in \mathcal{A}$. We have $U_\beta \subseteq \cup_{\alpha\in\mathcal{A}} U_\alpha$, so $\mathbb{R} - \cup_{\alpha\in\mathcal{A}} U_\alpha \subseteq \mathbb{R} - U_\beta$. The latter is finite, hence so is $\mathbb{R} - \cup_{\alpha\in\mathcal{A}} U_\alpha$, so $\cup_{\alpha\in\mathcal{A}} U_\alpha \in \mathcal{U}$. [Or one can use DeMorgan here: $\mathbb{R} - \cup_{\alpha\in\mathcal{A}} U_\alpha = \cap_{\alpha\in\mathcal{A}}(\mathbb{R} - U_\alpha) \subseteq \mathbb{R} - U_\beta$ is finite.]

Suppose that $\{U_i\}_{i=1}^n \subseteq \mathcal{U}$. If some $U_k = \emptyset$, then $\cap_{i=1}^n U_i = \emptyset \in \mathcal{U}$. Otherwise, each $\mathbb{R} - U_i$ is finite, and $\mathbb{R} - \cap_{i=1}^n U_i = \cup_{i=1}^n (\mathbb{R} - U_i)$ is a union of finite sets, so is finite. Therefore $\cap_{i=1}^n U_i \in \mathcal{U}$.

20. Define $\mathcal{A} = \{U \subset \mathbb{R} \,|\, U \text{ is finite}\} \cup \{\mathbb{R}\}$. Verify that $\mathcal{A}$ satisfies two of the three properties to be a topology on $\mathbb{R}$, but not the other one.

$\emptyset$ is finite, and $\mathbb{R} \in \mathcal{A}$ by definition.

Suppose that $\{U_i\}_{i=1}^n \subseteq \mathcal{A}$. If all $U_k = \mathbb{R}$, then then $\cap_{i=1}^n U_i = \mathbb{R} \in \mathcal{A}$. Otherwise, some U_k is finite, and $\cap_{i=1}^n U_i \subseteq U_k$ so $\cap_{i=1}^n U_i$ is finite and hence is in $\mathcal{A}$.

The union condition fails: For $n \in \mathbb{Z}$, $\{n\} \in \mathcal{A}$, but $\cup_{n\in\mathbb{Z}} \{n\} = \mathbb{Z} \notin \mathcal{A}$.

21. Define $\mathcal{A} = \{U \subset \mathbb{R} \,|\, \forall x \in U, \exists a, b \in \mathbb{R},\ \text{either } x \in [a,b) \subseteq U \text{ or } x \in (a,b] \subseteq U\}$. Verify that $\mathcal{A}$ satisfies two of the three properties to be a topology on $\mathbb{R}$, but not the other one.

$\emptyset \in \mathcal{A}$ since it vacuously satisfies the condition to be in $\mathcal{A}$. Let $x \in \mathbb{R}$. Then $x \in [x, x+1) \subseteq \mathbb{R}$, showing that $\mathbb{R} \in \mathcal{A}$.

Suppose that $\{U_\alpha\}_{\alpha\in\mathcal{A}} \subseteq \mathcal{A}$. Let $x \in \cup_{\alpha\in\mathcal{A}} U_\alpha$. Then $x \in U_\beta$ for some β. Since $U_\beta \in \mathcal{A}$, there exist $a, b \in \mathbb{R}$ such that either $x \in [a,b) \subseteq U_\beta$ or $x \in (a,b] \subseteq U_\beta$. Since $U_\beta \subseteq \cup_{\alpha\in\mathcal{A}} U_\alpha$, either $x \in [a,b) \subseteq \cup_{\alpha\in\mathcal{A}} U_\alpha$ or $x \in (a,b] \subseteq \cup_{\alpha\in\mathcal{A}} U_\alpha$.

The intersection condition fails: $[0,1)$ and $(-1,0]$ are in $\mathcal{A}$, but $[0,1) \cap (-1,0] = \{0\}$ is not in in $\mathcal{A}$. [To check this, if $0 \in [a,b)$ then $b > 0$ so $[a,b) \not\subseteq \{0\}$, and if $0 \in (a,b]$, then $a < 0$ so $(a,b] \not\subseteq \{0\}$.]

22. Let $X = \mathbb{Q}$, the set of rational numbers, with the subspace topology as a subset of $\mathbb{R}$. Prove that if U is any nonempty open subset of $\mathbb{Q}$, then there exist nonempty open subsets U_1 and U_2 such that $U = U_1 \cup U_2$ and $U_1 \cap U_2 = \emptyset$. Notice that this appears not to be true for the standard topology on $\mathbb{R}$ (and indeed it is not true).

Let U be a nonempty open set, and choose $x_0 \in U$. Now $U = V \cap \mathbb{Q}$ for some open set V in $\mathbb{R}$. Since V is open, there exists $\epsilon > 0$ such that $(x_0 - \epsilon, x_0 + \epsilon) \subset V$, and hence $(x_0 - \epsilon, x_0 + \epsilon) \cap \mathbb{Q} \subset U$. Choose a rational number r with $r \in (x_0, x_0 + \epsilon) \cap \mathbb{Q}$, and choose an irrational number s with $x_0 < s < r$. Let $V_1 = (-\infty, s) \cap \mathbb{Q}$ and $V_2 = (s, \infty) \cap \mathbb{Q}$. These are open in the subspace topology on $\mathbb{Q}$, and $V_1 \cup V_2 = (((-\infty, s) \cap \mathbb{Q}) \cup ((s, \infty) \cap \mathbb{Q}) = ((-\infty, s) \cup (s, \infty)) \cap \mathbb{Q} = (\mathbb{R} - \{s\}) \cap \mathbb{Q} = \mathbb{Q}$. For $i = 1, 2$, let $U_i = V_i \cap U$. Since $x_0 \in U_1$ and $r \in U_2$, U_1 and U_2 are nonempty. We also have $U_1 \cup U_2 = (V_1 \cup V_2) \cap U = \mathbb{Q} \cap U = U$, and $U_1 \cap U_2 \subseteq (-\infty, s) \cap (s, \infty) = \emptyset$.

23. Let X be a set and let $\mathcal{B}$ be a collection of subsets of X. Define $\mathcal{U} = \{U \subseteq X \mid \forall x \in U, \exists B \in \mathcal{B}, x \in B \subseteq U\}$. Prove that $U \in \mathcal{U}$ if and only if U is a union of elements of $\mathcal{B}$.

Suppose that $U \in \mathcal{U}$. For each $x \in U$, choose $B_x \in \mathcal{B}$ satisfying $x \in B_x \subseteq U$. Then $U \subseteq \cup_{x \in U}\{x\} \subseteq \cup_{x \in U} B_x \subseteq U$, so $U = \cup_{x \in U} B_x$.

Suppose that $U = \cup_{\gamma \in \mathcal{G}} B_\gamma$ where each $B_\gamma \in \mathcal{B}$. Let $x \in U$. Then $x \in B_\delta$ for some $\delta \in \mathcal{G}$, and $x \in B_\delta \subseteq U$.

24. Let X be a set and let $\mathcal{B}$ be a collection of subsets of X. We say that $\mathcal{B}$ is a *basis* if it satisfies:
B1. $X = \cup_{B \in \mathcal{B}} B$.
B2. $\forall B_1, B_2 \in \mathcal{B}, \forall x \in B_1 \cap B_2, \exists B \in \mathcal{B}, x \in B \subseteq B_1 \cap B_2$.
Define $\mathcal{U} = \{U \subseteq X \mid \forall x \in U, \exists B \in \mathcal{B}, x \in B \subseteq U\}$. Prove that if $\mathcal{B}$ is a basis, then $\mathcal{U}$ is a topology on X.

Since $X = \cup_{B \in \mathcal{B}} B$, Problem 23 shows that $X \in \mathcal{U}$. The empty set $\emptyset$ vacuously satisfies the condition $\forall x \in \emptyset, \exists B \in \mathcal{B}, x \in B \subseteq U$, so $\emptyset \in \mathcal{U}$.

Suppose $U_\alpha \in \mathcal{U}$ for $\alpha \in \mathcal{A}$. Let $x \in \cup_{\alpha \in \mathcal{A}} U_\alpha$. Then $x \in U_\beta$ for some $\beta \in \mathcal{A}$. There exists $B \in \mathcal{B}$ such that $x \in B \subseteq U_\beta \subseteq \cup_{\alpha \in \mathcal{A}}$. Therefore $\cup_{\alpha \in \mathcal{A}} \in \mathcal{U}$.

For T3, it suffices to show that the intersection of any two elements of $\mathcal{U}$ is in $\mathcal{U}$. Suppose that $U, V \in \mathcal{U}$, and that $x \in U \cap V$. Choose $B_U, B_V \in \mathcal{B}$ such that $x \in B_U \subseteq U$ and $x \in B_V \subseteq V$. There exists $B \in \mathcal{B}$ such that $x \in B \subseteq B_U \cap B_v \subseteq U \cap V$. Therefore $U \cap V \in \mathcal{B}$.

25. Let X be a topological space and let $\mathcal{B}$ be a basis for the topology on X. Let $A \subset X$, and define $\mathcal{B}_A = \{B \cap A \mid B \in \mathcal{B}\}$.
   a. Prove that $\mathcal{B}_A$ is a basis.
   b. Prove that $\mathcal{B}_A$ generates the subspace topology on A.

For B1, let $a \in A$. Since $\mathcal{B}$ is a basis, there exists $B_a \in \mathcal{B}$ such $a \in B_a$. Then, $a \in B_a \cap A \in \mathcal{B}_A$. Since each element of $\mathcal{B}_A$ is a subset of A, this proves that $A \subseteq \cup_{a \in A} B_a \subseteq \cup_{B \in \mathcal{B}_A} B \subseteq A$, so $A = \cup_{B \in \mathcal{B}_A} B$.

For B2, let $B_1, B_2 \in \mathcal{B}_A$, and let $a \in B_1 \cap B_2$. There exist basis elements $B_1', B_2' \in \mathcal{B}$ such that $B_1 = B_1' \cap A$ and $B_2 = B_2' \cap A$. Since $a \in B_1 \cap B_2 \subseteq B_1' \cap B_2'$, and $\mathcal{B}$ is a basis, there exists $B' \in \mathcal{B}$ such that $a \in B' \subseteq B_1' \cap B_2'$. So $a \in B' \cap A \subseteq (B_1' \cap B_2') \cap A = (B_1' \cap A) \cap (B_2' \cap A) = B_1 \cap B_2$.

To show that $\mathcal{B}_A$ generates the subspace topology, let $\mathcal{U}$ be the subspace topology and let $\mathcal{U}_A$ be the topology generated by $\mathcal{B}_A$.

Suppose that $U \in \mathcal{U}$. Let $a \in U$. There exists V open in X with $U = V \cap A$. Since $a \in V$ and $\mathcal{B}$ is a basis for the topology on X, there exists $B' \in \mathcal{B}$ such that $a \in B' \subseteq V$. So $a \in B' \cap A \subseteq V \cap A = U$, and $B' \cap A \in \mathcal{B}_A$. This proves that $\mathcal{U} \subseteq \mathcal{U}_A$.

Suppose that $U \in \mathcal{U}_A$. For every $a \in U$, choose $B_a \in \mathcal{B}_A$ with $x \in B_a \in U$. Each $B_a = B_a' \cap A$ for some $B_a' \in \mathcal{B}$. Let $V = \cup_{a \in A} B_a'$; it is a union of open sets in X, so is open in X. We have $V \cap A = (\cup_{a \in A} B_a') \cap A = \cup_{a \in A} (B_a' \cap A) = \cup_{a \in A} B_a = U$, so $U \in \mathcal{U}$.

We conclude that $\mathcal{U} = \mathcal{U}_A$.