Math 4853 homework

51. Let X be a set with the cofinite topology. Prove that every subspace of X has the cofinite topology (i. e. the subspace topology on each subset A equals the cofinite topology on A). Notice that this says that every subset of X is compact. So not every compact subset of X is closed (unless X is finite, in which case X and all of its subsets are finite sets with the discrete topology).

Let U be a nonempty open subset of A for the subspace topology. Then $U = V \cap A$ for some open subset V of X. Now $V = X - F$ for some finite subset F. So $V \cap A = (X - F) \cap A = (A \cap X) - (A \cap F) = A - (A \cap F)$. Since $A \cap F$ is finite, this is open in the cofinite topology on A. Conversely, suppose that U is an open subset for the cofinite topology of A. Then $U = A - F_1$ for some finite subset F_1 of A. Then, $X - F_1$ is open in X, and $(X - F_1) \cap A = A - F_1$, so $A - F_1$ is open in the subspace topology.

52. Let X be a Hausdorff space. Prove that every compact subset A of X is closed. Hint: Let $x \notin A$. For each $a \in A$, choose disjoint open subsets U_a and V_a of X such that $x \in U_a$ and $a \in V_a$. The collection $\{V_a \cap A\}$ is an open cover of A, so has a finite subcover $\{V_{a_i} \cap A\}_{i=1}^n$. Now, let $U = \cap_{i=1}^n U_{a_i}$, a neighborhood of x. Prove that $U \subset X - A$ (draw a picture!), which proves that $X - A$ is open.

To show that $X - A$ is open, it suffices to prove that for each $x \notin A$, there exists an open set U of X with $x \in U \subseteq X - A$. Fix some $x \in X - A$. For each $a \in A$, choose disjoint open subsets U_a and V_a of X such that $x \in U_a$ and $a \in V_a$. The collection $\{V_a \cap A\}$ is an open cover of A, so has a finite subcover $\{V_{a_i} \cap A\}_{i=1}^n$. Now, let $U = \cap_{i=1}^n U_{a_i}$, a neighborhood of x. We claim that $U \subseteq X - A$. Suppose for contradiction that $y \in U \cap A$. Since $y \in A$, $y \in V_{a_j}$ for some j. Also, $y \in U = \cap_{i=1}^n U_{a_i} \subseteq U_{a_j}$. This is impossible since U_{a_j} and V_{a_j} are disjoint.

53. Prove that the only connected nonempty subsets of $\mathbb{Q}$ are its one-point subsets.

We will prove that if S is a nonempty connected subset of Q, then S contains only one point, by arguing the contrapositive.

Suppose that S is a subset of $\mathbb{Q}$ that contains more than one point, say $a, b \in S$ with $a < b$. Choose an irrational number s with $a < s < b$. Let $U = (-\infty, s) \cap S$ and $V = (s, \infty) \cap S$. These are disjoint, open in S, nomempty (since $a \in U$ and $b \in V$), and their union is S (since $((-\infty, s) \cap S) \cup ((s, \infty) \cap S) = ((-\infty, s) \cup (s, \infty)) \cap S = (\mathbb{R} - \{s\}) \cap S = S$), so S is not connected.

[One may one to point out that a one-point subset *is* connected— this is immediate since a one-point set cannot admit a surjective function to $\{1, 2\}$.]

54. Let X be a topological space and let $S \subseteq X$. Prove that if S is connected, then $\overline{S}$ is connected.

Let $f\colon \overline{S} \to \{1,2\}$ be a continuous function. Then $f|_S$ is continuous. Since S is connected, $f|_S$ is constant, say $f(S) = 1$. Since f is continuous, $f(\overline{S}) \subseteq \overline{f(S)} = \overline{\{1\}} = \{1\}$, so f is also constant.

Alternatively, suppose for contradiction that $\overline{S} = U \cup V$ with U and V disjoint nonempty open subsets. Since S is connected, either $U \cap S$ or $V \cap S$ must be empty, say $V \cap S$ is empty. Then $S \subset U$. Now U is closed in $\overline{S}$, since its complement V is open, so $\overline{S} \subset U$. But then, V is empty, a contradiction.

55. Let X be an infinite set with the cofinite topology. Prove that X is connected.

Suppose that U and V were disjoint nonempty sets in X whose union is X. Now $U = X - F_U$ and $V - X - F_V$ for some finite subsets F_U and F_V. Since U and V are disjoint, $V \subset F_U$ and $U \subset F_V$. But then, $X = U \cup V \subseteq F_V \cup F_U$ would be finite, a contradiction.

56. Suppose A and B are connected subsets of a space X. Prove that if $A \cap B$ is nonempty, then $A \cup B$ is connected.

Suppose that $f\colon A \cup B \to \{1,2\}$ is continuous. Since A is connected, $f|_A$ is constant, say $f(A) \subseteq \{1\}$. Similarly, $f|_B$ is constant, so $f(A) \subseteq \{1\}$ or $f(B) \subseteq \{2\}$. The latter is impossible, since there is a point $x_0 \in A \cap B$ with $f(x_0) = 1$, so $f(B) \subseteq \{1\}$ and therefore $f(A \cup B) \subseteq \{1\}$. Since every function from $A \cup B$ to $\{1,2\}$ is constant, $A \cup B$ is connected.

Alternatively, suppose that $A \cup B = U \cup V$ with U and V disjoint open sets. Since A is connected, $A \subseteq U$ or $A \subseteq V$, say $A \subseteq U$. Similarly, $B \subseteq U$ or $B \subseteq V$. The latter is impossible, since there is a point $x_0 \in A \cap B$ with $x_0 \in U$, so $A \cup B \subseteq U$. Therefore V must be empty.

57. Let $X = C([0,1])$, the set of continuous functions from $[0,1]$ to $\mathbb{R}$. Define $\rho(f,g) = \max_{x\in[0,1]}\{(1-x^2)|f(x)-g(x)|\}$. Verify the triangle inequality. Hint: If $F(x) \leq G(x)$ for all $x \in [0,1]$, then $F(x) \leq \max_{x\in[0,1]}\{G(x)\}$ for all $x \in [0,1]$, and therefore $\max_{x\in[0,1]}\{F(x)\} \leq \max_{x\in[0,1]}\{G(x)\}$.

let $f, g, h \in X$. For all $x \in [0,1]$, we have

$$(1-x^2)|f(x)-g(x)| \leq (1-x^2)(|f(x)-h(x)||h(x)-g(x)|)$$
$$= (1-x^2)|f(x)-h(x)| + (1-x^2)|h(x)-g(x)| \leq \rho(f,h) + \rho(h,f)$$

and therefore $\rho(f,g) \leq \rho(f,h) + \rho(h,f)$.

58. Let (X,d) be a metric space and let A be a subset of X.

(a) Use continuity of d to prove that if A is compact, then A is closed. Hint: If A is not closed, there is a limit point z of A that is not contained in A. Consider the function $f\colon A \to \mathbb{R}$ defined by $f(x) = 1/d(x,z)$.

(b) Prove that X is Hausdorff, and apply Problem 52 to prove if A is compact, then A is closed.

For (a), suppose that A is not closed. Then there exists a limit point z of A that is not contained on A. Since d is continuous and $d(x, z) > 0$ for all $x \in A$, the function $f\colon A \to \mathbb{R}$ defined by $f(x) = 1/d(x, z)$ is continuous. To show that A is not compact, it suffices to show that f is unbounded. Let $M \in \mathbb{R}$ be any positive number. Since z is a limit point of A, there exists a point $x \in A \cap B(z, 1/M)$. Since $d(x, z) < 1/M$, $f(x) = 1/d(x, z) > M$. Therefore f is unbounded.

For (b), let $x, y \in X$. We will show that $B(x, d(x, y)/2)$ and $B(y, d(x, y)/2)$ are disjoint neighborhoods of x and y. If not, then there exists $z \in B(x, d(x, y)/2) \cap B(y, d(x, y)/2)$. But then, $d(x, y) \leq d(x, z) + d(z, y) \leq d(x, y)/2 + d(x, y)/2 = d(x, y)$, a contradiction. We conclude that X is Hausdorff. By Problem 52, compact subsets of X are closed.

59. Let $f\colon X \to Y$ be continuous. Suppose $\{ x_n \}$ is a sequence in X that converges to x. Prove that $\{ f(x_n) \}$ converges to $f(x)$.

Let U be any neighborhood of $f(x)$. Then $f^{-1}(U)$ is a neighborhood of x, so there exists N such that if $n \geq N$ then $x_n \in f^{-1}(U)$. That is, if $n \geq N$, then $f(x_n) \in U$.

60. Let X be a Hausdorff space. Prove that limits in X are unique. That is, if $\{ x_n \}$ is a sequence in X and $x_n \to x$ and $x_n \to y$, then $x = y$.

Suppose for contradiction that $x_n \to x$ and $x_n \to y$, but $x \neq y$. Choose disjoint neighborhoods U of x and V of y. There exist N_1 and N_2 such that if $n \geq N_1$ then $x_n \in U$, and if $n \geq N_2$ then $x_n \in V$. So if $n \geq \max\{N_1, N_2\}$, $x_n \in U \cap V$, contradicting the fact that U and V are disjoint.