Math 4853 homework

29. (3/12) Let X be a topological space. Suppose that $\mathcal{B}$ is a collection of subsets of X such that

    (1) The sets in $\mathcal{B}$ are open in X.

    (2) For every open set U in X and every $x \in U$, there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U$.

(a) Prove that $\mathcal{B}$ is a basis.

(b) Prove that the topology $\{V \subseteq X \mid \forall x \in V, \exists B \in \mathcal{B}, x \in B \subseteq V\}$ generated by $\mathcal{B}$ equals the topology of X.

30. (3/12) Use the previous problem to give a quick proof that if X is a topological space, A is a subset of X, and $\mathcal{B}$ is a basis for the topology of X, then $\{B \cap A \mid B \in \mathcal{B}\}$ is a basis for the subspace topology on A.

31. (3/12) Let $f\colon X \to Y$ and $g\colon Y \to Z$ be functions between topological spaces.

(a) Check that for any subset $V \subseteq Z$, $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$.

(b) Prove that if f and g are continuous, then $g \circ f$ is continuous.

(c) Give an example of non-continuous functions $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ whose composition is continuous.

32. (3/12) Let $f\colon X \to Y$ be a function between topological spaces, and let $\mathcal{B}$ be a basis for the topology on Y. Prove that if $f^{-1}(B)$ is open for every $B \in \mathcal{B}$, then f is continuous.

33. (not to turn in) Check any of the properties $f^{-1}(\cup A_\alpha) = \cup f^{-1}(A_\alpha)$, $f^{-1}(\cap A_\alpha) = \cap f^{-1}(A_\alpha)$, and $f(\cup A_\alpha) = \cup f(A_\alpha)$ that are not clear to you. Give a counterexample to $f(A \cap B) = f(A) \cap f(B)$.

34. (3/12) Complete the proof that if $f\colon (\mathbb{R}, \mathcal{S}) \to (\mathbb{R}, \mathcal{L})$ is continuous, then f is constant by completing Case II (assume that $x_1 < x_2$ and $f(x_1) > f(x_2)$ and reach a contradiction). Do it analogously to our method for Case I but using the greatest lower bound property— every nonempty set of real numbers that has a lower bound has a greatest lower bound. (There is also a simple trick for deducing Case II from Case I, but the point here is to practice this type of argument. If you want, try to also find the simple trick method.)

35. (4/2) Let $f\colon X \to Y$ be a function between topological spaces. Let Z be a subset of Y such that $f(X) \subseteq Z$, and define $g\colon X \to Z$ by $g(x) = f(x)$. We say that g is obtained from f by restriction of the codomain. Assuming, of course, that Z has the subspace topology as a subspace of Y, prove that f is continuous if and only if g is. (Moral of the story: It's OK to be careless about codomains.)