Math 4853 homework

36. (4/2) Prove that the continuous bijection $f\colon [0, 2\pi) \to S^1$ defined by $f(t) = (\cos(t), \sin(t))$ is not a homeomorphism.

37. (4/2) (a) Let $\mathcal{C}$ be the cofinite topology on $\mathbb{R}$. Prove that if $f^{-1}(\{r\})$ finite for every $r \in \mathbb{R}$, then $f\colon (\mathbb{R}, \mathcal{C}) \to (\mathbb{R}, \mathcal{C})$ is continuous. [You will need to use the fact that if $g\colon X \to Y$ is a function and $S \subseteq Y$, then $g^{-1}(Y - S) = X - g^{-1}(S)$, which you should check if it is not clear to you.]
(b) Examine the converse of part (a).

38. (not to turn in) Work through the following step-by-step argument proving that the following are equivalent for a bijection $\phi\colon X \to Y$ between topological spaces:

    (i) ϕ is a homeomorphism.

    (ii) U is open in X if and only if $\phi(U)$ is open in Y.

First, we observe that if $\phi\colon X \to Y$ is a bijection, and $A \subseteq X$, then $(\phi^{-1})^{-1}(A) = \phi(A)$. For we have $x \in (\phi^{-1})^{-1}(U) \Leftrightarrow \phi^{-1}(x) \in U \Leftrightarrow x = \phi(\phi^{-1}(x)) \in \phi(U)$.

Assume (i). Suppose first that U is open in X. Then $\phi(U) = (\phi^{-1})^{-1}(U)$ is open in Y, since ϕ^{-1} is continuous. Suppose now that $\phi(U)$ is open in Y. Then $U = \phi^{-1}(\phi(U))$ is open in X, since ϕ is continuous.

Now assume (ii). Let V be open in Y. Since $V = \phi(\phi^{-1}(V))$, (ii) implies that $\phi^{-1}(V)$ is open in X, so ϕ is continuous. Let U be open in X. Then $(\phi^{-1})^{-1}(U) = \phi(U)$, which is open by (ii). Therefore ϕ^{-1} is continuous.

39. (4/2) In $\mathbb{R}$, let $\mathcal{S} = \{(a, \infty) \mid a \in \mathbb{R}\} \cup \{(-\infty, b) \mid b \in \mathbb{R}\}$. Prove that $\mathcal{S}$ is a sub-basis that generates the standard topology. Find a similar sub-basis for the lower-limit topology.

40. (4/2) Let $\mathcal{B}_X$ and $\mathcal{B}_Y$ be bases for the topologies of two spaces X and Y. Prove that $\{B_1 \times B_2 \mid B_1 \in \mathcal{B}_X, B_2 \in \mathcal{B}_Y\}$ is a basis for the product topology on $X \times Y$.

41. (4/2) Let $f\colon X \to Y$ be a function. Recall that the *graph* of f is the subset $\Gamma_f \subset X \times Y$ defined by $\Gamma_f = \{(x, y) \mid f(x) = y\}$. Assume that X is a topological space and that Y is a *Hausdorff* topological space, which means that if y_1 and y_2 are any two points of Y, then there are *disjoint* open sets V_1 and V_2 in Y with $y_1 \in V_1$ and $y_2 \in V_2$. Prove that if f is continuous, then the complement $X \times Y - \Gamma_f$ of Γ_f is an open subset of $X \times Y$. Hint: Write $W = X \times Y - \Gamma_f$. It suffices to show that if $(x_0, y_0) \in W$, then there is a basic open set W' with $(x_0, y_0) \in W' \subseteq W$. Since $(x_0, y_0) \in W$, $f(x_0) \neq y_0$ and therefore there are disjoint open sets V_1 and V_2 in Y with $f(x_0) \in V_1$ and $y_0 \in V_2$. Now examine $W' = f^{-1}(V_1) \times V_2$ (draw a picture!).